Effective coupling for open billiards

Konstantin Pichugin
Institute of Physics, Czech Academy of Sciences, Cukrovarnicka 10, Prague, Czech Republic and Kirensky Institute of Physics, 660036 Krasnoyarsk, Russia

Holger Schanz*
Max-Planck-Institut für Strömungsforschung und Institut für Nichtlineare Dynamik der Universität Göttingen, Bunsenstraße 10, D-37073 Göttingen, Germany

Petr Šeba
Institute of Physics, Czech Academy of Sciences, Cukrovarnicka 10, Prague, Czech Republic and Department of Physics, Pedagogical University, Hradec Kralove, Czech Republic

(Received 9 April 2001; published 26 October 2001)

We derive an explicit expression for the coupling constants of individual eigenstates of a closed billiard that is opened by attaching a waveguide. The Wigner time delay and the resonance positions resulting from the coupling constants are compared to an exact numerical calculation. Deviations can be attributed to evanescent modes in the waveguide and to the finite number of eigenstates taken into account. The influence of the shape of the billiard and of the boundary conditions at the mouth of the waveguide are also discussed. Finally we show that the mean value of the dimensionless coupling constants tends to the critical value when the eigenstates of the billiard follow random-matrix theory.

DOI: 10.1103/PhysRevE.64.056227 PACS number(s): 05.45.Mt, 03.65.Nk, 72.20.Dp

I. INTRODUCTION

During the last years quantum chaotic scattering was a field of intense research. A great deal of the results obtained was based on the projection operator formalism due to Feshbach, Weidenmüller, and others [1–3]. In this approach the scattering system is decomposed into a closed subsystem described by the internal Hamiltonian $H_{\rm in}$ with discrete bound states $n=1,\ldots,N$ and a continuum of external scattering states labeled by the energy E and an index $\lambda=1,\ldots,\Lambda(E)$ corresponding to different open scattering channels. The coupling between the internal and external subsystems is then incorporated by an operator with matrix elements $W_{n,\lambda}(E)$. The S matrix of the complete system can be expressed in terms of these matrix elements and the Hamiltonian $H_{\rm in}$. This relation can be cast into the form

$$S=\frac{I-iK}{I+iK}, \tag{1}$$

$$K=\pi W^{\dagger}\frac{I}{E-H_{\rm in}}W. \tag{2}$$

Here, S and K are energy-dependent square matrices of dimension $\Lambda\times\Lambda$, and W has dimension $N\times\Lambda$. While this setting is very general, the tools developed for the subsequent analysis of the properties of the S matrix require additional assumptions. In particular, the energy thresholds for the opening of new scattering channels are usually neglected. As a consequence the energy dependence of the coupling matrix $W(E)$ can be considered weak and then the S matrix (1) can be rewritten in terms of an $N\times N$ effective non-Hermitian Hamiltonian $H_{\rm eff}$,

$$S=I-2\pi i W^{\dagger}\frac{I}{E-H_{\rm eff}}W, \tag{3}$$

$$H_{\rm eff}=H_{\rm in}-i\pi WW^{\dagger}. \tag{4}$$

We will refer to this canonical formalism [1–3] for expressing the S matrix as the *Hamiltonian approach* to scattering. We use this name to distinguish it from an S matrix obtained directly, i.e., without reference to any auxiliary closed system and its Hamiltonian.

When the internal Hamiltonian in Eq. (4) describes a chaotic system, it is justified to replace it by a random matrix [4], and by performing an average over the appropriate ensemble a statistical theory for the S matrix is obtained that allows to calculate quantities of interest such as correlation functions or the distribution of Wigner delay times and resonance poles [5–8]. It was found that the results of such an approach are to a large extent independent of the detailed structure of the matrix W, but they do depend on the dimensionless mean coupling strength

$$g=\pi^2\frac{\langle|W_{n,\lambda}|^2\rangle_{n,\lambda}}{D}. \tag{5}$$

Here, D is the mean energy level spacing of the internal subsystem, and the average $\langle\cdots\rangle_{n,\lambda}$ is taken over the internal states n and all open scattering channels λ.

For example, when the coupling constant (5) exceeds the critical value $g=1$ and the number of scattering channels is small compared to the total number of states, a counter-

*Electronic address: holger@chaos.gwdg.de

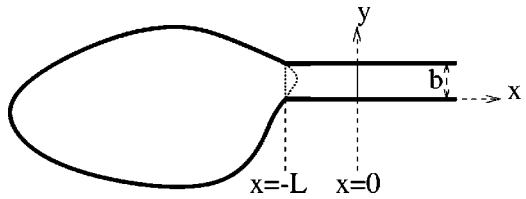

FIG. 1. A scattering system consisting of an infinite waveguide and a cavity is shown with bold lines. Various possibilities to add a wall and obtain a closed billiard are shown with thin lines (solid and dotted).

intuitive shrinking of the widths of most resonances with increasing coupling is observed [9–13]. For each attached scattering channel only one of the resonance widths grows further with the coupling g. The resulting redistribution of S-matrix poles was coined resonance trapping.

Chaotic billiards with attached waveguides are considered as paradigm for chaotic scattering [14]. They are relevant as theoretical models for understanding the transport properties of mesoscopic semiconductor structures [15] or experimental results on microwave scattering in flat resonators [16–19]. Also signatures of resonance trapping were recently observed in billiards, both numerically [20–22] and in microwave resonator experiments [19].

However, to our knowledge there is no theory that maps a given billiard to an effective Hamiltonian with overcritical coupling, thus really establishing a connection between the numerically and experimentally observed phenomena and the results on resonance trapping obtained within the formalism (1)–(5).

Motivated by this situation, it is the purpose of the present paper to discuss the application of the Hamiltonian approach to open billiards in some detail and to answer questions such as: How can Eqs. (1) and (3) be derived for a billiard, what kind of approximations are involved and what is the resulting expression for the coupling constants $W_{n,\lambda}(E)$? What is the influence of the choice of the internal subsystem that is *not unique* for a given scattering system?

Using the expression for the coupling constants $W_{n,\lambda}$ to be derived in Sec. II we will then address the effective coupling constant for a typical chaotic billiard with an attached waveguide. We show in Sec. III that *in the semiclassical regime*, and when no tunneling barriers obstruct the waveguides, the coupling strength is fixed at the critical value $g=1$, independently of the size or the precise geometry of the billiard and of its openings. A numerical verification of our results is contained in Sec. IV, followed by a short discussion on the implications of our findings.

II. COUPLING CONSTANTS FOR INDIVIDUAL LEVELS

We consider a situation as shown in Fig. 1. A scattering system is formed in two dimensions by an infinite waveguide of width b and and an arbitrary cavity. Inside the system the potential is identically 0. We set $\hbar=2m=1$ and $E=k^2$ such that the stationary Schrödinger equation reduces to the Helmholtz equation

$$(\Delta+k^2)\Psi(x,y)=0, \tag{6}$$

with $\Delta=\partial^2/\partial_x^2+\partial^2/\partial_y^2$. On the boundary of the scattering system (bold solid line in Fig. 1) we require Dirichlet boundary conditions (BC) $\Psi=0$. This boundary condition and also the precise geometry of the system are by no means essential, the following generalizes, e.g., immediately to a cavity with more than one attached lead or Neumann BC $\partial/\partial_n\Psi=0$. In Fig. 1 we have shown several possibilities to define a closed billiard that corresponds to the scattering system in question (solid and dotted thin lines). We will restrict the discussion to the case shown with a solid line. We require that the boundary of the internal system is located inside the attached waveguide and that it consists of a transversal straight line on which either Dirichlet or Neumann boundary conditions are imposed to close the system. Clearly, even this restriction makes the correspondence between the scattering system and the auxiliary internal system not unique because the exact position of the closure along the waveguide is variable. We use coordinates where this closure is at $x=0$ while the matching between waveguide and cavity is at $x=-L$ $(L>0)$.

In the region of the attached waveguide $(x\geqslant -L)$ we can decompose any function into transversal modes

$$\phi_\lambda(y)=\sqrt{2/b}\sin(\lambda\pi y/b) \quad (\lambda=1,2,\ldots), \tag{7}$$

because these functions form a complete and orthonormal basis on the interval $(0,b)$ according to

$$\sum_{\lambda=1}^{\infty}\phi_\lambda(y)\phi_\lambda(y')=\delta(y-y') \quad (0<y,y'<b) \tag{8}$$

and

$$\int_0^b dy\,\phi_\lambda(y)\phi_{\lambda'}(y)=\delta_{\lambda\lambda'}. \tag{9}$$

The most general solution of the Helmholtz equation is a superposition of scattering states $\Psi_\lambda(x,y)$. They consist of a single incoming wave in transversal mode λ and the corresponding outgoing modes given by the S matrix of the system

$$\Psi_\lambda(x,y)=\phi_\lambda(y)\frac{e^{-ik_\lambda x}}{\sqrt{k_\lambda}}+\sum_{\lambda'}S_{\lambda'\lambda}\phi_{\lambda'}(y)\frac{e^{+ik_{\lambda'}x}}{\sqrt{k_{\lambda'}}}. \tag{10}$$

The longitudinal wave number $k_\lambda=\sqrt{k^2-(\lambda\pi/b)^2}$ is real for $\lambda\leqslant\Lambda=[kb/\pi]$, where $[\cdots]$ denotes the integer part. These modes are called open or travelling, and the $\Lambda\times\Lambda$ matrix $S_{\lambda'\lambda}$ corresponding to the open modes is the unitary S matrix we are interested in. For $\lambda>\Lambda$ the momentum along the waveguide is imaginary. These modes are called closed or evanescent. In the scattering state Ψ_λ with $\lambda\leqslant\Lambda$ the evanescent outgoing modes describe exponentially decaying contributions that modify the wave function in the vicinity of the mouth of the waveguide. Evanescent incoming modes are exponentially increasing into the waveguide and thus unphysical for the scattering system.

When the evanescent modes are included, the S matrix becomes an infinite-dimensional operator that is no longer unitary. It is possible to construct an eigenstate of the closed billiard by a superposition of the scattering states (10) including evanescent modes. Suppose $S(E)$ has an eigenvalue unity at some energy $E_n=k_n^2$ and let $a_{n,\lambda}$ be the components of the corresponding eigenvector. Then the linear combination of scattering states

$$\Psi_n^{(\mathrm{N})}(x,y)=\sum_\lambda \frac{a_{n,\lambda}^{(\mathrm{N})}}{2}\Psi_\lambda(x,y)=\sum_\lambda \frac{a_{n,\lambda}^{(\mathrm{N})}}{\sqrt{k_{n,\lambda}}}\phi_\lambda(y)\cos(k_{n,\lambda}x) \tag{11}$$

$(-L\leqslant x\leqslant 0)$ satisfies Neumann BC at $x=0$ and it is thus indeed an eigenfunction of the billiard because it satisfies Eq. (6) and the remaining boundary conditions by construction. The normalization of the S-matrix eigenvector $a_{n,\lambda}$ in Eq. (11) is *not* unity but rather determined by the normalization of the billiard eigenfunction $\Psi_n^{(\mathrm{N})}(x,y)$. If Dirichlet boundary conditions are required at $x=0$ the same argument can be repeated for an eigenvalue -1 of the S matrix and we have

$$\Psi_n^{(\mathrm{D})}(x,y)=\sum_\lambda \frac{a_{n,\lambda}^{(\mathrm{D})}}{2i}\Psi_\lambda(x,y)=\sum_\lambda \frac{a_{n,\lambda}^{(\mathrm{D})}}{\sqrt{k_{n,\lambda}}}\phi_\lambda(y)\sin(k_{n,\lambda}x) \tag{12}$$

$(-L\leqslant x\leqslant 0)$. Consequently the spectrum of the billiard closed with Neumann or Dirichlet BC can be found from the secular equation

$$\det[I\mp S(E)]=0, \tag{13}$$

which was first derived by Doron and Smilansky [23]. In a sense we will in the following invert this so-called scattering approach to the quantization of billiards. We will express the S matrix in terms of the eigenvalues and eigenfunctions of the closed system.

For this purpose consider the Green function of the closed billiard that is defined as the resolvent of $-\Delta$ in the space of functions that satisfy the boundary conditions of the billiard. In position representation this definition can be expressed by the inhomogeneous Helmholtz equation

$$(\Delta+k^2)G(\mathbf{r};\mathbf{r}',k)=\delta(\mathbf{r}-\mathbf{r}'). \tag{14}$$

In the eigenbasis of the billiard the Green function reads

$$G(\mathbf{r};\mathbf{r}',k)=\sum_{n=1}^{\infty}\frac{\Psi_n^*(\mathbf{r}')\Psi_n(\mathbf{r})}{k^2-k_n^2}, \tag{15}$$

which can be verified using $\Delta\Psi_n(\mathbf{r})=-k_n^2\Psi_n(\mathbf{r})$ and the completeness of the functions Ψ_n inside the billiard. For $\mathbf{r}$ and $\mathbf{r}'$ inside the waveguide we can expand the Green function with respect to the transversal modes $\phi_\lambda(y)$ and find as the general form of a solution of Eq. (14),

$$G(\mathbf{r},\mathbf{r}',k)=\frac{1}{2i}\sum_{\lambda\lambda'}\frac{\phi_\lambda(y)}{\sqrt{k_\lambda}}\frac{\phi_{\lambda'}(y')}{\sqrt{k_{\lambda'}}}\times\Bigg[\delta_{\lambda\lambda'}e^{ik_\lambda|x-x'|}+\sum_{s,s'=\pm}G_{\lambda\lambda'}^{ss'}(k)\exp(isk_\lambda x+is'k_{\lambda'}x')\Bigg]. \tag{16}$$

Indeed, the first term inside the brackets gives rise to one particular solution of the inhomogeneous equation (14), while the second term with the unknown matrices G^{++}, G^{+-}, G^{-+}, and G^{--} represents the most general solution of the corresponding homogeneous Helmholtz equation (6). The unknown coefficients must be determined such that the Green function satisfies also the boundary conditions inside the cavity and on the transversal closure of the waveguide. For this purpose assume first $x\geqslant x'$ and consider $\mathbf{r}$ as a fixed parameter. Then $G(\mathbf{r};\mathbf{r}',k)$ as a function of $\mathbf{r}'$ should satisfy the homogeneous Helmholtz equation with the boundary conditions of the scattering system, i.e., it can be written as a superposition of the scattering states $\Psi_\lambda(\mathbf{r}')$ defined in Eq. (10). On the other hand, when $\mathbf{r}'$ is fixed, the Green function as a function of $\mathbf{r}$ satisfies the boundary conditions (Neumann or Dirichlet) at $x=0$ where the closed billiard is separated from the waveguide by the additional straight wall. Thus, it must be a superposition of the functions

$$\Psi_\lambda^{\mathrm{N/D}}(\mathbf{r})=\frac{\phi_\lambda(y)}{\sqrt{k_\lambda}}(e^{ik_\lambda x}\pm e^{-ik_\lambda x}), \tag{17}$$

which are in fact the scattering states for a semi-infinite waveguide with Neumann or Dirichlet BC at one end. Consequently, the Green function has the form

$$G(\mathbf{r};\mathbf{r}',k)=\frac{1}{2i}\sum_{\lambda,\lambda'}\Psi_\lambda^{\mathrm{N/D}}(\mathbf{r})g_{\lambda\lambda'}(k)\Psi_{\lambda'}(\mathbf{r}'), \tag{18}$$

with another set of undetermined coefficients $g_{\lambda\lambda'}$. Expanding Eq. (18) into transversal modes and comparing to Eq. (16) we obtain

$$G^{++}=gS,\quad G^{+-}=g-I,\qquad G^{-+}=\pm gS,\quad G^{--}=\pm g. \tag{19}$$

We can now repeat this argumentation under the opposite assumption $x<x'$ and find again the relations (19) but with G^{+-} and G^{-+} exchanged. This can be regarded as a consequence of the symmetry of the Green function with respect to its two arguments. This symmetry, in turn, follows from time-reversal symmetry. We conclude $G^{+-}=G^{-+}=g-I=\pm gS$ and hence

$$g(k)=[I\mp S(k)]^{-1}. \tag{20}$$

Note that $g(k)$ and thus the Green function diverges as expected at the solutions of the secular equation (13), i.e., when k corresponds to an eigenvalue of the closed billiard.

Using Eq. (20) we can now directly relate the S matrix to the transversal expansion coefficients of the Green function at the closure of the billiard. We define for x,x' inside the waveguide

$$G_{\lambda,\lambda'}(x,x')=\int_0^b dy\,dy'\,\phi_\lambda(y)G(x,y;x',y';k)\phi_{\lambda'}(y') \tag{21}$$

and

$$K^{(\mathrm{N})}_{\lambda\lambda'}=\sqrt{k_\lambda k_{\lambda'}}G_{\lambda,\lambda'}(0,0), \tag{22}$$

$$K^{(\mathrm{D})}_{\lambda\lambda'}=\frac{1}{\sqrt{k_\lambda k_{\lambda'}}}\frac{\partial^2}{\partial_x\partial_{x'}}G_{\lambda,\lambda'}(x,x')\big|_{x=x'=0} \tag{23}$$

and derive from Eq. (18) using Eq. (20)

$$iK^{(\mathrm{N})}=\frac{I+S}{I-S}\quad(\text{Neumann BC at } x=0), \tag{24}$$

$$iK^{(\mathrm{D})}=\frac{I-S}{I+S}\quad(\text{Dirichlet BC at } x=0). \tag{25}$$

Obviously, $K^{(\mathrm{N})}=0$ for Dirichlet BC and $K^{(\mathrm{D})}=0$ for Neumann BC at $x=0$. Eqs. (24) and (25) can be inverted and yield

$$S=-\frac{I-iK^{(\mathrm{N})}}{I+iK^{(\mathrm{N})}}=+\frac{I-iK^{(\mathrm{D})}}{I+iK^{(\mathrm{D})}}, \tag{26}$$

which is now in the form of Eq. (1) (for Neumann BC up to an irrelevant constant phase). We can now proceed to determine the corresponding coupling constants $W^{(\mathrm{N/D})}_{n,\lambda}$ by representing the K matrix in the eigenbasis of the billiard. From Eq. (2) we have

$$K_{\lambda\lambda'}=\pi\sum_{n=1}^{\infty}\frac{W^*_{n,\lambda}W_{n,\lambda'}}{k^2-k_n^2} \tag{27}$$

and from Eqs. (15), (22), and (23) we find

$$K^{(\mathrm{N})}_{\lambda\lambda'}=\sqrt{k_\lambda k_{\lambda'}}\sum_{n=1}^{\infty}\frac{\Psi^{(\mathrm{N})*}_{n,\lambda}(0)\Psi^{(\mathrm{N})}_{n,\lambda'}(0)}{k^2-k_n^2}, \tag{28}$$

$$K^{(\mathrm{D})}_{\lambda\lambda'}=\frac{1}{\sqrt{k_\lambda k_{\lambda'}}}\sum_{n=1}^{\infty}\frac{\frac{\partial}{\partial x}\Psi^{(\mathrm{D})*}_{n,\lambda}(0)\frac{\partial}{\partial x}\Psi^{(\mathrm{D})}_{n,\lambda'}(0)}{k^2-k_n^2}, \tag{29}$$

where we have introduced the projections

$$\Psi^{(\mathrm{N/D})}_{n,\lambda}(x)=\int_0^b dy\,\phi_\lambda(y)\Psi^{(\mathrm{N/D})}_n(x,y) \tag{30}$$

of the eigenfunctions of the closed billiard onto the transversal modes of the waveguide. The values of the coupling constants follow from comparing Eq. (27) to Eqs. (28) and (29),

$$W^{(\mathrm{N})}_{n,\lambda}=\sqrt{\frac{k_\lambda}{\pi}}\Psi^{(\mathrm{N})}_{n,\lambda}(0), \tag{31}$$

$$W^{(\mathrm{D})}_{n,\lambda}=\frac{1}{\sqrt{k_\lambda\pi}}\frac{\partial}{\partial x}\Psi^{(\mathrm{D})}_{n,\lambda}(0). \tag{32}$$

This form of the dependence of the coupling constants on the internal wave functions is not surprising; also within perturbation theory the coupling depends on the value of the wave function at the point where the system is opened or on its normal derivative for Neumann and Dirichlet boundary conditions, respectively. However, in the situation we consider perturbation theory is not applicable and, in particular, the precise value of the prefactor in the coupling constants (31) and (32) could only be obtained from the derivation given in this section.

The representation (26) of the S matrix in terms of the K matrices (28) and (29) is exact, when all transversal modes are included. However, usually one is interested only in the $\Lambda\times\Lambda$ unitary part of the S matrix, and this is only *approximately* given by Eq. (26) when the K matrix is restricted to open modes. In [23,24] the effect of this so-called semiquantal approximation for the accuracy of eigenvalues within the scattering approach to quantization was investigated numerically. It becomes negligible when the energy E is sufficiently far from the threshold for the opening of a new channel. Under this restriction we can consider $W^{(\mathrm{N/D})}_{n,\lambda}$ as the coupling constants corresponding to the unitary part of the S matrix. When E approaches a threshold, Eq. (3) breaks down, since the energy dependence of the coupling constants can no longer be neglected. We will not consider this case here.

III. THE MEAN COUPLING STRENGTH

Given the explicit values (31) and (32) for the coupling constants between individual states and individual scattering channels we are now going to derive an estimate for the dimensionless coupling strength (5) in the semiclassical limit and neglecting evanescent modes. Since the concept of a mean coupling strength is not well defined for infinitely many internal states, the internal Hamiltonian H_{in} entering Eqs. (2) and (4) should for this purpose be cut to some finite matrix including only states that are close enough in energy $k_n\sim k$. In particular this means that we can replace the momenta along the waveguide $k_{n,\lambda}$ in Eqs. (11) and (12) by their on-shell values k_λ. The resulting approximate expansions of the billiard eigenfunctions are projected onto the transversal modes according to Eq. (30) and inserted into Eqs. (31) and (32) that simplify to

$$W^{(\mathrm{N/D})}_{n,\lambda}=\frac{a^{(\mathrm{N/D})}_{n,\lambda}}{\sqrt{\pi}}. \tag{33}$$

At this point it is necessary to determine the average magnitude of the coefficients $a^{(\mathrm{N/D})}_{n,\lambda}$. We assume that the classical dynamics of the billiard is chaotic. In the semiclassical limit this means that the quantum ergodicity theorem applies to the eigenstates of the billiard, i.e., in particular the probabil-

ity density integrated over an arbitrary region of the billiard tends to the relative area of that region. Applied to the part of the billiard inside the waveguide we find

$$\frac{bL}{A} \approx \int_0^b dy \int_{-L}^0 dx |\Psi_n^{(\mathrm{N})}(x,y)|^2 = \int_{-L}^0 dx \sum_\lambda \frac{|a_{n,\lambda}|^2}{k_\lambda} \cos^2(k_\lambda x) \approx \langle |a|^2 \rangle \frac{L}{2} \int_0^{kb/\pi} d\lambda \sqrt{\frac{1}{k^2 - (\lambda\pi/b)^2}} = \langle |a|^2 \rangle \frac{bL}{4}, \tag{34}$$

where A denotes the total area of billiard. In the second line we have inserted the normal mode decomposition into $|\Psi_n^{(\mathrm{N})}(x,y)|^2$. The orthonormalization of the transversal modes was then used to restrict the resulting double sum over modes to diagonal terms. In the third line $\langle \cos^2 \rangle = 1/2$ was used ($\langle \sin^2 \rangle = 1/2$ in the completely analogous calculation for Dirichlet BC), and the sum over modes was approximated by a continuous integral. This is justified when the number of modes is large, i.e., in the semiclassical limit. Using the resulting constraint on the normalization of the coefficients of the billiard eigenfunctions in the transversal basis implied by Eq. (34), $\langle |a_{n,\lambda}^{(\mathrm{N/D})}|^2 \rangle = 4/A$, we find

$$\langle |W_{n,\lambda}^{(\mathrm{N/D})}|^2 \rangle = \frac{4}{A\pi}. \tag{35}$$

According to Eq. (5) the average coupling between the internal states and the continuum must be normalized by the mean level spacing D of the billiard that is the only independent energy scale of the system. To leading semiclassical order we have Weyl's law $D = 4\pi/A$ [25] that finally results in

$$g = 1. \tag{36}$$

IV. NUMERICAL RESULTS

To check the validity of the Hamiltonian approach to scattering in a quantum billiard we have performed direct numerical calculations for a Sinai billiard connected to a single waveguide (see Fig. 2). First, we evaluated numerically the 1300 lowest eigenvalues and eigenvectors of the closed system with Dirichlet and Neumann BC at the boundary segment to which the waveguide was attached. Using the expressions (30)–(32) we calculated the elements of the coupling matrix $W_{n,\lambda}^{(\mathrm{N/D})}$.

Knowing W and using Eqs. (1) and (2) we evaluated—as the next step—the S matrix and compared it to the S matrix obtained by a direct method based on the numerical solution of the underlying Schrödinger equation (see Ref. [26] for details). In order to visualize the differences between these two S matrices we compared first the corresponding Wigner-Smith time delays

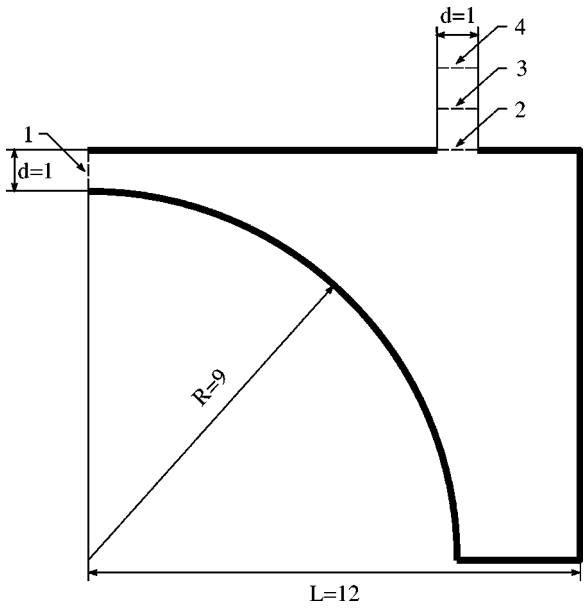

FIG. 2. Geometry of the billiard. Four different possibilities to attach the waveguide are used.

$$\tau(E) = \frac{i}{M} \mathrm{Tr}\left(\frac{\partial S^\dagger(E)}{\partial E} S(E) \right), \tag{37}$$

where M is the number of the open channels inside the waveguide. The results are plotted in Fig. 3.

The figure shows an outstanding agreement of the two approaches for low energies. The Hamiltonian approach describes the positions as well as the widths of the narrow resonances with high accuracy. The agreement is good even close to the threshold energies of the individual channels. For higher energies, however, the difference between the two time delay functions increases due to the limited number of internal states included into the evaluation of the Hamiltonian approach S matrix. Similar results (not displayed) were obtained also with the waveguide attached to the

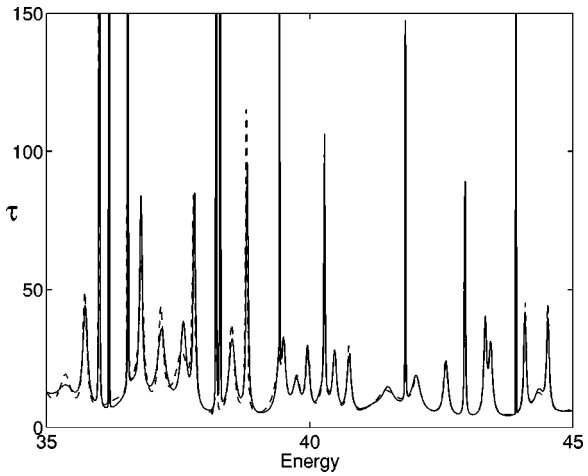

FIG. 3. The Wigner-Smith time delay obtained for the Hamiltonian approach S matrix with Neumann BC (dashed line) compared with the result of a direct evaluation (solid line). In the displayed case the waveguide was attached to the boundary No. 1.

boundaries No. 2, 3, and 4, respectively. In all these cases the Neumann matching procedure was used.

On the other hand, for Dirichlet matching equation (32) the results change drastically. In this case the agreement is not good even for low energies. This may seem surprising because the derivations of the previous sections were entirely parallel for Neumann and Dirichlet BC. However, an important difference is hidden in the convergence properties of the spectral decompositions of the K matrix (28) and (29) as we shall explain now. Projecting Eqs. (11) and (12) onto transversal mode λ we find,

$$\Psi_{n,\lambda}^{(\mathrm{N})}(0)=\frac{a_{n,\lambda}^{(\mathrm{N})}}{\sqrt{k_{n,\lambda}}},\quad \frac{\partial}{\partial x}\Psi_{n,\lambda}^{(\mathrm{D})}(0)=\sqrt{k_{n,\lambda}}a_{n,\lambda}^{(\mathrm{D})}. \tag{38}$$

Since the coefficients $a_{n,\lambda}^{(\mathrm{N/D})}$ are according to Eq. (34) of order $2/\sqrt{A}$ that is independent of n, we have from Eqs. (31) and (32),

$$W_{n,\lambda}^{(\mathrm{N})}\sim\sqrt{\frac{k_\lambda}{k_{n,\lambda}}},\quad W_{n,\lambda}^{(\mathrm{D})}\sim\sqrt{\frac{k_{n,\lambda}}{k_\lambda}}. \tag{39}$$

For a two-dimensional billiard $k_n, k_{n,\lambda}=O(\sqrt{n})$ $(n\to\infty)$ such that the terms in the infinite spectral sum (27) decay asymptotically as $n^{-3/2}$ for Neumann and as $n^{-1/2}$ for Dirichlet BC. Hence the convergence is absolute for Neumann BC while Eq. (29) converges at most conditionally. As a consequence, the numerically necessary cut off in the summation over the internal states n introduces large errors for Dirichlet BC.

For the following considerations we will concentrate on Neumann BC. As already mentioned, the choice of the internal and external parts of the system is not unique, since an arbitrary part of the ideal waveguide can be considered part of the internal system. Increasing the length of the waveguide included, we decrease in fact the influence of the evanescent modes since they are exponentially vanishing inside waveguide. We have checked this relation and evaluated the time delay functions also for various waveguide parts included into the internal system. The results remain practically unchanged regardless of the length of the included part. This demonstrates the small influence of the evanescent modes on the resulting S matrix.

Knowing the coupling matrix W and using the relation Eq. (5) we evaluated numerically the value of the coupling constant g. The obtained result is in an excellent agreement with the estimated value (36) for all considered types of the waveguide attachment leading to $g\approx 0.98$. We have evaluated the coupling constant g also for a different shape of the billiard [20] obtaining similar values for g. It has to be stressed that the estimate (36) was obtained using semiclassical arguments. Our calculation shows, however, that it remains valid even in the deep quantum region.

The eigenvalues of H_{eff} equation (4) are usually interpreted as the resonance poles and are used for the study of the statistical properties of resonances in open quantum chaotic systems [7]. To check the validity of this approach we have evaluated the resonance poles of the system independently using the complex scaling method [22] that provides a direct access to the positions of the poles of the analytically continued S matrix. The obtained results were compared with the eigenvalues of the effective Hamiltonian (4). However, a direct comparison is obscured by the fact that the coupling matrix W is in fact energy dependent. In the standard random matrix approach the coupling matrix is treated as being energy independent—a simplification that is well justified inside a small energy interval. To mimic this situation and to minimize the influence of the energy dependence of the coupling matrix W we have compared the eigenvalues of H_{eff} with the directly evaluated resonance poles always within a small energy interval the center of which was equal to the energy used to evaluate the coupling matrix W.

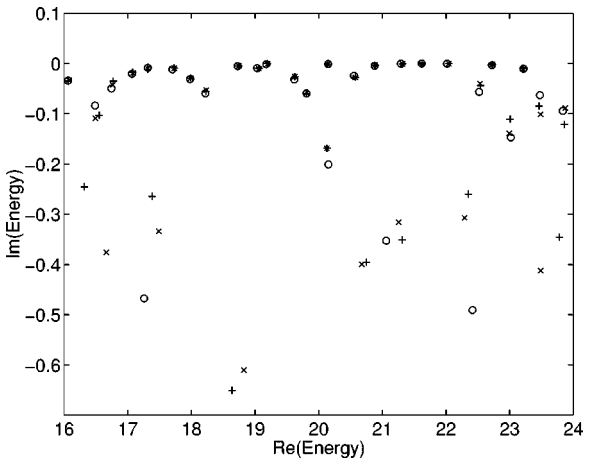

FIG. 4. The complex eigenvalues of H_{eff} $(+)$, zeros of $I+iK^{(\mathrm{N})}$ $(\times)$, and resonance poles obtained by the complex scaling method $(\bigcirc)$.

The energy dependence of W can be taken into account more precisely using the relation (1) and evaluating the Hamiltonian approach resonance poles as zeros of the function $I+iK^{(\mathrm{N})}(E)$. We have evaluated the resonance poles using both of the above described methods and compared the results to the numerically exact resonances obtained by complex scaling. The results are shown in Fig. 4. From this figure we see that for narrow resonances the eigenvalues of effective Hamiltonian H_{eff} represent a good approximation to the resonance poles of the system and the energy dependence of the coupling matrix W can be omitted. For broad resonances the situation changes and the complex eigenvalues of the effective Hamiltonian have nothing in common with the directly evaluated resonance poles. This discrepancy can be explained as follows: A resonance localized at $E_R=E+i\Gamma$ represents in fact a collective mode of all bound states E_n of the internal Hamiltonian H_{in} that are located inside the energy interval $\approx(E-\Gamma,E+\Gamma)$. For a broad resonance with Γ significantly larger than the mean spacing between the bound states E_n the number of the internal states to be included into H_{eff} must therefore be very high.

V. CONCLUSIONS

To summarize, we have shown that the Hamiltonian approach to scattering, which is the basis of many important

random-matrix results on quantum chaotic scattering, leads to reasonably good agreement when compared with the results of a direct calculation of the S matrix in the case of billiards with Neumann boundary conditions. We have explained the somewhat unexpected finding that the accuracy is much worse for Dirichlet boundary conditions, while it does not depend very much on other possibilities of varying the auxiliary closed system used, such as the position of the attached waveguide.

Even for Neumann boundary conditions the effective Hamiltonian H_{eff} based on the Heidelberg approach seems to be not very well suited for the computation of broad resonances of the system. This result does not contradict the fact, that also for systems with time-reversal symmetry the *statistical* properties of billiard resonances follow the predictions of random-matrix theory based upon the effective Hamiltonian approach quite well [27], because our test goes way beyond a purely statistical analysis.

Moreover, we have shown numerically and with semiclassical arguments that the mean dimensionless coupling for a chaotic billiard is the critical value $g=1$—irrespective of the precise form of the billiard, the size of the attached waveguide, and other details of the model. Interestingly, in our model an effective coupling near $g=1$ is observed already deep in the quantum regime. Nevertheless, fluctuations around the mean value $g=1$ should in general be largest for small energies and can possibly result locally in overcritical coupling. This might be an explanation for the observed resonance trapping in billiards [19–22].

Our results concerning the value of the effective coupling for chaotic systems are not restricted to billiards and apply, e.g., to quantum graphs as well. In these systems, a systematic way to achieve overcritical coupling for many states is to modulate the density of states, e.g., by considering systems with band spectra [3,28].

ACKNOWLEDGMENTS

Stimulating discussions with F. Dittes and D. Savin are gratefully acknowledged. This work has been partially supported by Czech Grant No. GAAV A1048804 and by the ''Foundation for Theoretical Physics'' in Slemeno, Czech Republic. K.P. acknowledges financial support from grant RFFI No. 01-02-16077.

[1] C. Mahaux and H.A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).
[2] H. Feshbach, *Theoretical Nuclear Physics* (Wiley-Interscience, New York, 1992).
[3] F.M. Dittes, Phys. Rep. **339**, 216 (2000).
[4] O. Bohigas, M.J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[5] P.A. Mello, P. Pereyra, and T.H. Seligman, Ann. Phys. (N.Y.) **161**, 254 (1985).
[6] C.H. Lewenkopf and H.A. Weidenmüller, Ann. Phys. (N.Y.) **212**, 53 (1991).
[7] Y.V. Fyodorov and H.J. Sommers, J. Math. Phys. **38**, 1918 (1997).
[8] T. Guhr, A. Müller-Groeling, and H.A. Weidenmüller, Phys. Rep. **299**, 190 (1998).
[9] P.A. Moldauer. Phys. Rev. Lett. **18**, 249 (1967); Phys. Rev. **171**, 1164 (1968); Phys. Rev. C **11**, 426 (1975).
[10] J.J.M. Verbaarschot, H.A, Weidenmüller, and M.R. Zirnbauer, Phys. Rep. **29**, 367 (1985).
[11] V.V. Sokolov and V.G. Zelevinsky, Nucl. Phys. A **504**, 562 (1989).
[12] V.V. Sokolov and V.G. Zelevinsky, Ann. Phys. (N.Y.) **216**, 323 (1992).
[13] I. Rotter, Rep. Prog. Phys. **54**, 635 (1991).
[14] E. Doron and U. Smilansky, Phys. Rev. Lett. **68**, 1255 (1992).
[15] C.M. Marcus, A.J. Rimberg, R.M. Westervelt, P.F. Hopkins, and A.C. Gossard, Phys. Rev. Lett. **69**, 506 (1992).
[16] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).
[17] E. Doron, U. Smilansky, and A. Frenkel, Phys. Rev. Lett. **65**, 3072 (1990).
[18] H. Alt *et al.*, Nucl. Phys. A **560**, 293 (1993).
[19] E. Persson, I. Rotter, H.J. Stöckmann, and M. Barth, Phys. Rev. Lett. **85**, 2478 (2000).
[20] E. Persson, K. Pichugin, I. Rotter, and P. Šeba, Phys. Rev. E **58**, 8001 (1998).
[21] I. Rotter, E. Persson, K. Pichugin, and P. Šeba, Phys. Rev. E **62**, 450 (2000).
[22] P. Šeba, I. Rotter, M. Müller, E. Persson, and K. Pichugin, Phys. Rev. E **61**, 66 (2000).
[23] E. Doron and U. Smilansky, Nonlinearity **5**, 1055 (1992).
[24] H. Schanz and U. Smilansky, Chaos, Solitons Fractals **5**, 1289 (1995).
[25] H.P. Baltes and E.R. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut, Mannheim, 1976).
[26] T. Ando, Phys. Rev. B **44**, 8017 (1991).
[27] H. Ishio, Phys. Rev. E **62**, R3035 (2000).
[28] H. Schanz and F. M. Dittes (unpublished).