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Quantum decay of an open chaotic system: A semiclassical approach

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Abstract. – We study the quantum probability to survive in an open chaotic system in the framework of the van Vleck-Gutzwiller propagator and present the first such calculation that accounts for quantum interference effects. Specifically, we calculate quantum deviations from the classical decay after the break time t^* for both broken and preserved time-reversal symmetry. The source of these corrections is identified in interfering pairs of correlated classical trajectories. In our approach the quantized chaotic system is modelled by a quantum graph.

A fundamental source of physical information are time-resolved decay measurements in open quantum-mechanical systems. While the radioactive decay is a prominent paradigm, more recent experiments studied atoms in optically generated lattices and billiards [1–6], the ionization of molecular Rydberg states [7] and excitaton relaxation in semiconductor quantum dots and wires [8, 9]. Most of these examples, and also the complementary theoretical investigations of quantum decay [10–20], address the semiclassical regime of systems with chaotic classical limit.

However, despite this broad interest there is no satisfactory semiclassical theory for the observed quantum dynamics. It is known from numerical studies and random-matrix theory (RMT) calculations [16–19] that the quantum survival probability $P(t)$ follows the exponential classical decay $P_{\rm cl}(t)$ only up to a break time t^*. This break time scales with the number of open decay channels L and the classical lifetime $t_{\rm cl}$ as $t^* \sim \sqrt{L}\, t_{\rm cl}$ [16]. For $t > t^*$, the quantum decay law is a universal function which depends only on L, $t_{\rm cl}$ and the Heisenberg time $t_{\rm H}$ and is qualitatively different from $P_{\rm cl}(t)$ [17–19]. Up to now, none of these results was accessible by semiclassical calculations and thus their applicability to individual chaotic systems remained a matter of speculation.

In this letter we show that a systematic semiclassical expansion for the quantum decay can be based on the van Vleck-Gutzwiller propagator. Specifically, we obtain with this approach the above-mentioned features for the quantum probability to survive inside a quantized network (quantum graph), which is one of the standard models in quantum chaos [21, 22]. We identify the source of quantum deviations from the classical decay in the interference between certain pairs of correlated classical trajectories (inset of fig. 1). We have calculated the

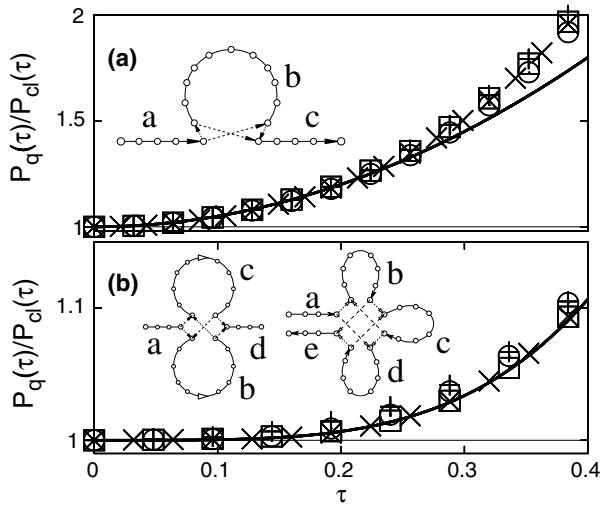

Fig. 1 – The ratio between quantum and classical survival probability has been computed for a quantum graph with $L = 10$ attached decay channels and $G = 500(+)$, $750(\times)$, $1000(\square)$, $2000(\circ)$ internal states. It is shown as a function of the scaled time $\tau = t/G$. Quantum corrections are visible as deviations of the data from the horizontal line. For small τ they follow (a) eq. (18) in the case with time-reversal symmetry and (b) eq. (19) without. These semiclassical predictions account for the interference within the pairs of classical trajectories shown schematically in the insets. The two trajectories forming a pair are identical along the segments $a, b, \ldots$, but differ in the crossing regions (solid *vs.* dashed arrows).

quantum corrections analytically to leading order in time for both, broken and preserved time-reversal symmetry (TRS). The resulting expressions, eqs. (18), (19) below, are in convincing agreement with the corresponding numerical data (fig. 1).

Although we study in the present paper only systems with universal behaviour, the semiclassical approach which we develop bears the potential to include system-specific properties as well. It is natural that this important advantage over any *ad hoc* random-matrix assumption comes at the price that *semiclassical calculations cannot be completely independent of the underlying model*. Therefore, some technical details of our calculation are specific to quantized networks. Nevertheless, our results are of interest also beyond this class of models because the same pairs of correlated trajectories will give rise to quantum corrections also in other chaotic systems. This expectation is based on the analogy to weak-localization corrections in spectral and transport properties [23–28].

It must be stressed, however, that the quantum corrections which we are describing here go beyond the known weak-localization effects as their presence is not restricted to systems with time-reversal symmetry. In previous studies of pairs of interfering classical trajectories it was always found that they have no net effect on two-point correlation functions if time-reversal symmetry is broken. This applies to both, open and closed systems [23–28]. Corrections have been found in the case of shot noise [29], but this quantity is a higher-order correlator and involves groups of four correlated trajectories. So, besides the new physical context in which we study the effect of classical action correlations, the novelty in the present work lies in the fact that they can have a non-vanishing effect on two-point correlators even without time-reversal symmetry.

The dynamics of a quantum system is governed by a time-evolution operator U propagating

an initial state $|\psi_0\rangle$ from time 0 to t as

$$|\psi_t\rangle = U^t\,|\psi_0\rangle. \tag{1}$$

For a closed system U is unitary and therefore the norm of the initial state is preserved, $||\psi_t||^2 \equiv 1$. In contrast, for an open system U is sub-unitary. In this case, the norm of the state, *i.e.*, the survival probability

$$||\psi_t||^2 = \langle\psi_t|\psi_t\rangle = \big\langle\psi_0|U^{t\dagger}U^t|\psi_0\big\rangle, \tag{2}$$

decays from unity to zero as a function of time. We describe the smooth part of this decay. Superimposed fluctuations are removed by considering an average $P(t) = \langle||\psi_t||^2\rangle_{\psi_0,k}$ over the initial state $|\psi_0\rangle$. An additional average is over the energy k. Assuming that the dynamically relevant energy window contains G states ($G \to \infty$ in the semiclassical regime) and choosing a discrete basis $|m\rangle$, the average over $|\psi_0\rangle$ leads to

$$P(t) = G^{-1}\big\langle \operatorname{tr} U^t U^{t\dagger}\big\rangle_k = G^{-1}\sum_{m,n}\big\langle|(U^t)_{mn}|^2\big\rangle_k\,. \tag{3}$$

U can be approximated semiclassically by the van Vleck-Gutzwiller propagator [30,31],

$$\big(U^t\big)_{mn} = \sum_p \mathcal{A}_p\,\exp[\mathrm{i}\,R_p(m,n;t)/\hbar], \tag{4}$$

i.e., a matrix element describing the transition $n \to m$ has contributions from all classical trajectories p leading from m to n in time t. R_p is the action of the trajectory and $\mathcal{A}_p$ a complex amplitude combining a stability factor and an additional phase from the Maslov index of p. With eq. (4) we obtain for the survival probability

$$P(t) = G^{-1}\left\langle \sum_{p,q}\mathcal{A}_p\mathcal{A}_q^*\,\exp\left[\mathrm{i}\,[R_p - R_q]/\hbar\right]\right\rangle, \tag{5}$$

where the summation is over all pairs of classical trajectories p, q which start and end at the same point and have not left the system up to time t.

The classical probability to remain inside the system is reproduced if eq. (5) is restricted to the "diagonal" terms $p = q$. Then $P(t)$ is simply a sum over all possible trajectories p with corresponding probabilities $|\mathcal{A}_p|^2$. It decays exponentially as

$$P_{\mathrm{cl}}(t) \sim \exp[-t/t_{\mathrm{cl}}], \tag{6}$$

provided that the ergodic time of the chaotic flow is much shorter than the decay time t_{cl} that is obtained from the relative phase-space area of the opening [32].

To justify the diagonal approximation, which was developed originally in the context of the spectral two-point correlator [33], one observes that in the semiclassical limit $\hbar \to 0$ the exponential in eq. (5) represents rapidly oscillating phases which cancel upon averaging unless $R_{p,q}$ are correlated. This is certainly the case for $p = q$, but recent work on action correlations [23,26,28,29,34,35] has shown that other pairs contribute as well. The correlations in the actions stem from the fact that these trajectories are composed of long segments where they follow each other closely. In short crossing regions the order and/or orientation of the segments is modified to yield different trajectories $p \neq q$. It is convenient to represent trajectories by a symbolic code in which letters $a, b, \ldots$ stand for a whole segment. The codes of p

and q are then different words composed of the same set of letters. The number of segments, *i.e.*, the length of the code word, determines the relative importance of the action correlations.

Here we apply this scheme to the survival probability. As the trajectories p, q must start and end at the same points, their symbolic codes start and end with the same letter. We need then at least three segments to obtain different trajectories:

$$p = [abc], \qquad q = \left[a\hat{b}c\right], \tag{7}$$

where $\hat{b}$ denotes the time-reversal of b (see inset of fig. 1a). These are precisely the pairs which are responsible for the weak-localization contribution to the conductance [23] and they are present for TRS only. A contribution which is present for systems with and without TRS involves at least a four-letter word:

$$p = [abcd], \qquad q = [acbd] \tag{8}$$

(left inset of fig. 1b). However, it will be shown below that the contributions from all such orbit pairs cancel. Therefore, in order to capture the leading-order quantum corrections for systems without TRS, we need to consider trajectory pairs with five segments of the form

$$p = [abcde], \qquad q = [adcbe] \tag{9}$$

(right inset of fig. 1b). All other permutations with a, e fixed are excluded, since two segments can be combined into a single one such that a pair of the form (8) results.

The following calculation will be based on the trajectory pairs (7)-(9). It is plausible that they give the leading-order quantum corrections because the two partner trajectories deviate only in a minimum number of permutation points. This assertion is supported by our final result and also by related work on spectral correlations and transport [23, 26, 28, 29].

The next step is to perform the summation in eq. (5). For this purpose, we need explicit expressions for the amplitudes and the actions of the classical trajectories and therefore we consider a specific model system. We assume a quantized network as it is commonly used, *e.g.*, in mesoscopic physics [36] and quantum chaos [21, 26, 37]. The discrete time-evolution operator on a network with B directed bonds has the form of a product of two $B \times B$ matrices,

$$U(k) = S \times D(k). \tag{10}$$

Here $D_{mn}(k) = \delta_{mn}\, \mathrm{e}^{\mathrm{i}kl_m}$ is a diagonal matrix containing phase factors which describe the free propagation along the bonds of the network. l_m ($m = 1, \ldots, B$) denotes the bond lengths which are chosen to be incommensurate in order to avoid non-generic degeneracies. The wave number k will be used for averaging, $\langle\cdot\rangle_k = \lim_{k\to\infty} k^{-1} \int_0^k \mathrm{d}k' \,(\cdot)$. In particular, we have

$$\left\langle \mathrm{e}^{\mathrm{i}k(l_m - l_n)} \right\rangle_k = \delta_{mn}\,. \tag{11}$$

The matrix S fixes the topology of the underlying graph and the classical transition probabilities between its bonds, $P_{n\to m} = |S_{mn}|^2$. $P_{n\to m}$ specifies a Markovian random walk on the graph which is the classical analogue of eq. (10). We make the simplifying assumption that all transitions have equal probability,

$$|S_{mn}|^2 = B^{-1}. \tag{12}$$

The phases of S are still to be determined. We consider first a closed system. Then S must be unitary. As a second condition we note that for preserved time-reversal symmetry the

time-evolution operator is a symmetric matrix up to a unitary transformation [38]. This is satisfied if $S = S^{\mathrm{T}}$, since then $D^{\frac{1}{2}}\, U\, D^{-\frac{1}{2}}$ is symmetric. These two assumptions together with eq. (12) lead to a natural choice for the matrix elements, $S^{(1)}_{mn} = B^{-1/2} \exp[2\pi \mathrm{i} mn/B]$ [37]. In order to break time-reversal invariance, we have to destroy the symmetry of S but at the same time preserve the unitarity and eq. (12). This is achieved by a simple transformation exchanging neighboring rows of S, namely $S^{(2)} = \Lambda S^{(2)}$ with $\Lambda_{mn} = \delta_{m,n-(-1)^n}$. We have checked numerically that, according to spectral statistics, a closed graph with $S^{(1,2)}$ is a generic model for quantum chaos in the presence or absence of TRS, respectively (see also [26, 37]).

Finally, we need to open the system. A standard way to do this is to restrict the time evolution to a subset $\mathcal{G}$ of $G < B$ internal states by the projector $\Pi^{(\mathcal{G})}$ with non-zero matrix elements $\Pi^{(\mathcal{G})}_{mn} = 1$ for $m = n \leq G$,

$$\tilde{U} = \Pi^{(\mathcal{G})}\, U\, \Pi^{(\mathcal{G})}. \tag{13}$$

Effectively, the remaining $L = B - G$ bonds are then perfectly absorbing and play the role of attached decay channels (leads). Nevertheless, the set of these bonds $\mathcal{L}$ influences the dynamics on $\mathcal{G}$, *e.g.*, via the identity

$$\sum_{m\in\mathcal{G}} S_{mn} S^*_{mn'} = \delta_{nn'} - \sum_{m\in\mathcal{L}} S_{mn} S^*_{mn'} \quad \forall n, n' \tag{14}$$

expressing the unitarity $SS^\dagger = I$ of the closed system.

The Heisenberg time of a network model is given by the number of bonds, $t_{\mathrm{H}} = B$ [21]. As our final results scale with t_{H}, it will be convenient to represent them in terms of the scaled time $\tau \equiv t/B$. Whenever τ is used, we imply the semiclassical limit $t_{\mathrm{H}} \to \infty$. This limit is taken with τ and L fixed and we will keep only the leading-order terms in $t_{\mathrm{H}} = B$.

Substituting eq. (10) into eq. (4) and expanding $(U^t)_{mn}$, one finds that a trajectory p of length t on the graph is just a sequence of bonds $p_0, \ldots, p_t$ with $p_0 = n$ and $p_t = m$. Amplitude and phase are given by $\mathcal{A}_p = S_{p_t p_{t-1}} \ldots S_{p_1 p_0}$ and $R_p/\hbar = k(l_{p_0} + \cdots + l_{p_t})$, respectively [21]. A considerable simplification results from the fact that due to eq. (11) only orbit pairs with equal total lengths survive. Consequently, the phase factor in eq. (5) is absent.

For $p = q$ we obtain $P_{\mathrm{cl}}(t) = \sum_p |\mathcal{A}_p|^2 = (G/B)^t = (1 - L/B)^t$, *i.e.*, $P_{\mathrm{cl}}(\tau) = \exp[-L\tau]$. This is equivalent to eq. (6) with $t_{\mathrm{cl}} = B/L$. For the summation over p, we have used that according to eq. (12) the total probability of a trajectory of length t is B^{-t}, and that there are G^{t+1} such trajectories as each bond p_i is summed over the whole set $\mathcal{G}$.

In trajectory pairs composed of s segments, the total time t is the sum of the lengths of the segments and of a contribution from each of the $s-1$ crossing points $t = s - 1 + t_a + t_b + \ldots$. Besides this constraint, $t_a, t_b, \ldots$ can take any value ≥ 1. This yields a combinatorial factor $\binom{t-s}{s-1} \sim (B\tau)^{s-1}/(s-1)!$. Within the segments, amplitudes pair to classical probabilities, eq. (12), as in the diagonal contribution above. Only the bonds right at the crossing points between segments must be treated separately as a phase difference between p and q occurs there. We denote the factor from all crossing points for the moment by Φ_{pq} and proceed first to the summation over the inner bonds of the segments. For a segment i of length t_i, containing $t_i - 1$ inner bonds, this yields $G^{-1}(G/B)^{t_i}$. The prefactor G^{-1} is absent for the first and the last segment as the first or last bond is not involved in phase factors and can be summed. Including the explicit factor G^{-1} form, eq. (5), we have thus $G^{1-s}(G/B)^{t-s+1} \to G^{1-s} \exp[-L\tau]$. Further, G^{1-s} cancels B^{s-1} in the combinatorial factor above to leading order in B. All terms together finally yield

$$P_{pq}(\tau) = \tau^{s-1}/(s-1)!\, \Phi_{pq}\, P_{\mathrm{cl}}(\tau)\,, \tag{15}$$

where the phases Φ_{pq} remain to be calculated. We have

$$\begin{aligned}\Phi_{a\hat{b}c} &= \sum S_{\beta\bar{\alpha}} S^*_{\bar{\beta}\bar{\alpha}} S_{\gamma\bar{\beta}} S^*_{\gamma\beta}\,,\\ \Phi_{acbd} &= \sum S_{\beta\bar{\alpha}} S^*_{\gamma\bar{\alpha}} S_{\gamma\bar{\beta}} S^*_{\beta\bar{\gamma}} S_{\delta\bar{\gamma}} S^*_{\delta\bar{\beta}}\,,\\ \Phi_{adcbe} &= \sum S_{\beta\bar{\alpha}} S^*_{\delta\bar{\alpha}} S_{\gamma\bar{\beta}} S^*_{\gamma\bar{\delta}} S_{\delta\bar{\gamma}} S^*_{\beta\bar{\gamma}} S_{\varepsilon\bar{\delta}} S^*_{\varepsilon\bar{\beta}}\,,\end{aligned} \tag{16}$$

where $\bar{\alpha}, \beta, \bar{\beta}, \ldots$ stand for the last bond in a, the first in b, the last in b, ... and the summation for each index extends over the open graph $\mathcal{G}$ up to some crucial restrictions which we discuss now. Consider a pair in the form (7) with $b = \beta b' \bar{\beta}$, assume $\beta = \bar{\beta}$ and define $a' = a\beta$ and $c' = \beta c$. Then the very same orbit pair can also be represented as $p = [a'b'c']$, $q = [a'\hat{b}'c']$. We exclude this ambiguity by imposing the restriction $\beta \neq \bar{\beta}$. Similarly, for the pairs (8) we impose $\beta \neq \gamma$, $\bar{\beta} \neq \bar{\gamma}$ and for the pairs (9) $\beta \neq \delta$, $\bar{\beta} \neq \bar{\delta}$. With the help of these restrictions and eq. (14) we have

$$\begin{aligned}\Phi_{a\hat{b}c} &= \sum_{\bar{\alpha}\gamma\in\mathcal{L}} \sum_{\beta\neq\bar{\beta}\in\mathcal{G}} S_{\beta\bar{\alpha}} S^*_{\bar{\beta}\bar{\alpha}} S_{\gamma\bar{\beta}} S^*_{\gamma\beta}\\ &= \sum_{\bar{\alpha}\gamma\in\mathcal{L}} \left(\sum_{\beta\bar{\beta}\in\mathcal{G}} - \sum_{\beta=\bar{\beta}\in\mathcal{G}} \right) S_{\beta\bar{\alpha}} S^*_{\bar{\beta}\bar{\alpha}} S_{\gamma\bar{\beta}} S^*_{\gamma\beta}\,.\end{aligned} \tag{17}$$

In the second term with $\beta = \bar{\beta}$, the amplitudes S combine into B^{-2} and thus this term yields L^2G/B^2, which is negligible for $B \to \infty$. In the first term we apply eq. (14) to perform the (unrestricted) summation over β, $\bar{\beta}$. We obtain $LG^2/B^2 \to L$ ($B \to \infty$) plus another negligible term. Thus from eq. (15) with $s = 3$ we obtain

$$P_{a\hat{b}c}(\tau) = (L/2)\,\tau^2\,P_{\rm cl}(\tau). \tag{18}$$

Applying repeatedly eq. (14) and estimating the order of the resulting terms for $B \to \infty$ we find further

$$\begin{aligned}P_{acbd}(\tau) &= 0,\\ P_{adcbe}(\tau) &= (L^2/24)\,\tau^4\,P_{\rm cl}(\tau).\end{aligned} \tag{19}$$

These expressions agree with the RMT [17,18] to leading order in τ. This can be attributed to the rapidly mixing dynamics of our model, see eq. (12). It is expected that higher orders are reproduced as well, if in addition to eqs. (7)-(9) trajectory pairs with more segments are included. Their omission is also responsible for the remaining deviations between eqs. (18), (19) and our numerical data (fig. 1). These deviations are larger for TRS, where additional pairs with $s = 4, 5$ exist, while further corrections independent of TRS have at least $s \geq 6$.

In conclusion, we have obtained semiclassical expressions for the leading-order quantum corrections which determine the dynamics of the survival probability after the break time t^*. As a corollary we are also able to calculate the break time itself within the semiclassical theory: Using the definition $c = P_{\rm q}(t^*)/P_{\rm cl}(t^*)$ (c is some constant threshold for relevant deviations from $P_{\rm cl}$) and substituting $P_{\rm q} = P_{\rm cl} + P_{a\hat{b}c}$ for TRS and $P_{\rm q} = P_{\rm cl} + P_{adcbe}$ for broken TRS, we get $t^* \sim B/\sqrt{L} = \sqrt{L}t_{\rm cl}$ which agrees with the break times evaluated numerically in [16].

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