Shot Noise in Chaotic Cavities from Action Correlations

Holger Schanz,* Mathias Puhlmann, and Theo Geisel
Max-Planck-Institut für Strömungsforschung und Institut für Nichtlineare Dynamik der Universität Göttingen, Bunsenstraße 10, D-37073 Göttingen, Germany
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We consider universal shot noise in ballistic chaotic cavities from a semiclassical point of view and show that it is due to action correlations within certain groups of classical trajectories. Using quantum graphs as a model system, we sum these trajectories analytically and find agreement with random-matrix theory. Unlike all action correlations which have been considered before, the correlations relevant for shot noise involve four trajectories and do not depend on the presence of any symmetry.

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One of the most prominent methods to describe spectral and transport properties of ballistic quantum systems with classically chaotic dynamics relies on the semiclassical representation of the Green's function in terms of classical trajectories. For closed systems, this leads to Gutzwiller's trace formula [1], which expresses the oscillating density of states as a sum over periodic orbits. For open systems, the semiclassical theory of chaotic scattering [2,3] and, in particular, its applications to electronic transport through mesoscopic devices [4] are based on this approach. The power of this method as compared, e.g., to random-matrix theory (RMT) [5,6] is its potential to account for system-specific details. Its main drawback lies in the difficulty to handle the resulting sums over huge sets of classical trajectories. Until recently the only method to deal with this problem was Berry's diagonal approximation [7], which neglects any nontrivial correlations between the trajectories. As a consequence, many interesting phenomena such as weak localization, universality in spectral statistics and in conductance fluctuations, or the suppression of shot noise cannot be described properly. Although the role of correlations between the actions of classical orbits has been appreciated for a long time [8–10], there is only one special case where they can be accounted for explicitly: Sieber and Richter recently calculated the leading order weak localization corrections from correlations between specific orbit pairs [11]. This result stimulated further intense research [12,13] and it is clearly a very promising approach. However, as mentioned above, weak localization is just one of a variety of other phenomena for which action correlations are relevant as well, but cannot be accounted for by the orbit pairs considered in [11–13].

In this Letter, we address shot noise in ballistic mesoscopic conductors, which is an important source of experimentally accessible information about the dynamics in such systems [14–17]. We identify the action correlations which are responsible for shot noise and explain how due to these correlations the universal result of RMT [5,14–21] can be recovered from a single system without ensemble average. The relevant correlations are fundamentally different from all those considered previously [8–13] because they involve four instead of just two classical trajectories. Moreover, they do not depend on the presence of symmetries. We emphasize that action correlations are no small correction. They are needed to understand shot noise to leading order.

We will perform all explicit calculations in a specific model system, the quantum graph of Fig. 1(b). Quantum graphs (networks) have a long record as models for electronic transport (see [22] and references therein). Since the pioneering work of Kottos and Smilansky, they are also established in quantum chaos [12,23–25]. They are particularly suitable for our purpose as the representation in terms of classical trajectories is exact and the analogue of action correlations amounts to exact degeneracies. Previous studies showed that despite these analytical simplifications the mechanism and the role of correlations between classical trajectories are equivalent to other systems such as billiards [12].

Consider a chaotic cavity with two attached waveguides supporting N_1, N_2 transversal modes, respectively [Fig. 1(a)]. For small bias voltage and temperature, negligible electron interactions, and fully coherent dynamics, all information about electron transport through this system is contained in the scattering matrix at the Fermi energy

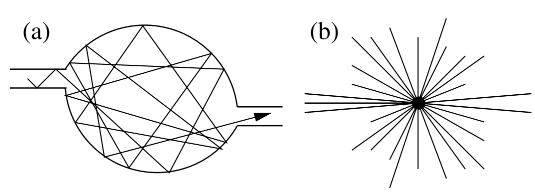

FIG. 1. (a) A chaotic cavity with two attached waveguides and a classical trajectory contributing to conductance and shot noise. (b) The quantum graph used to model this situation. The transversal modes of the waveguides correspond to the infinite leads attached to the graph (bold lines), while the internal system is represented by many finite bonds.

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \tag{1}$$

Shot noise represents temporal current fluctuations due to the discreteness of the electron charge [26]. At zero temperature, the average power of the noise can be expressed in terms of the transmission matrix t as [5,27]

$$P = 2e|V|G_0 \mathrm{Tr}\, tt^\dagger(1 - tt^\dagger), \tag{2}$$

while the assumption of uncorrelated electrons yields $P_{\mathrm{Poisson}} = 2e|V|G$. Here $G = G_0 \mathrm{Tr}\, tt^\dagger$ denotes the conductance, e is the electron charge, V the voltage, and $G_0 = 2e^2/h$ the conductance quantum. RMT yields [5]

$$\langle P\rangle_{\mathrm{RMT}} = 2e|V|G_0 N_1^2 N_2^2/(N_1 + N_2)^3, \tag{3}$$

and for the conductance $\langle G\rangle_{\mathrm{RMT}} = G_0 N_1 N_2/(N_1 + N_2)$. Each result comes with a weak localization correction which is small ($\langle \delta P\rangle \ll \langle P\rangle$, $\langle \delta G\rangle \ll \langle G\rangle$) for large N_1, N_2 and will not be considered here. For $N_1 = N_2$, the Fano factor $F = \langle P\rangle/\langle P_{\mathrm{Poisson}}\rangle = 1/4$ is obtained.

We will show that Eq. (3) can be recovered semiclassically. To this end, the element $t_{n_2 n_1}$ of the transmission matrix is expressed as a sum over all classical trajectories [28] connecting the incoming mode $1 \le n_1 \le N_1$ with the outgoing mode $1 \le n_2 \le N_2$ [29]

$$t_{n_2 n_1} = \sum_p A_p e^{\mathrm{i} S_p/\hbar}, \tag{4}$$

where S_p denotes the classical action and A_p is an amplitude related to the stability of the trajectory. While for the conductance we have to evaluate a sum over pairs of trajectories

$$\langle \mathrm{Tr}\, tt^\dagger\rangle = \sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2}\sum_{pq:n_1\to n_2} A_p A_q^* \langle e^{\mathrm{i}/\hbar(S_p - S_q)}\rangle, \tag{5}$$

the shot noise involves also a term combining four classical paths:

$$\langle \mathrm{Tr}\,(tt^\dagger)^2\rangle = \sum_{m_1 n_1}\sum_{m_2 n_2}\sum_{pqrs} A_p A_q^* A_r A_s^* \times \langle e^{\mathrm{i}/\hbar(S_p - S_q + S_r - S_s)}\rangle. \tag{6}$$

Here, the trajectories p, q, r, s connect two incoming to two outgoing modes as shown in Fig. 2(a). As Eqs. (5) and (6) describe one particular system rather than an ensemble, the average $\langle\cdot\rangle$ is to be taken over an energy window. It should be small enough to keep the classical dynamics and the amplitudes A_p essentially unchanged. Nevertheless, in the semiclassical limit $\hbar \to 0$, the phase factor is rapidly oscillating and only those orbit combinations survive the averaging for which the action changes are correlated.

In particular, setting $p = q$ in Eq. (5), the phase drops out and we are left with a sum over classical probabilities $|A_p|^2$. This is the diagonal approximation [7]. Provided that the dwell time inside the open cavity is large compared to the time needed for equidistribution over the available phase space, the probability is the same for all outgoing modes and $\langle G\rangle_{\mathrm{RMT}}$ is exactly recovered [4]. Hence, the contribution from other pairs of correlated trajectories that might exist must vanish although no explicit demonstration of this fact has been given until now. In the presence of time-reversal symmetry, the above remains valid to leading order in the mode number N.

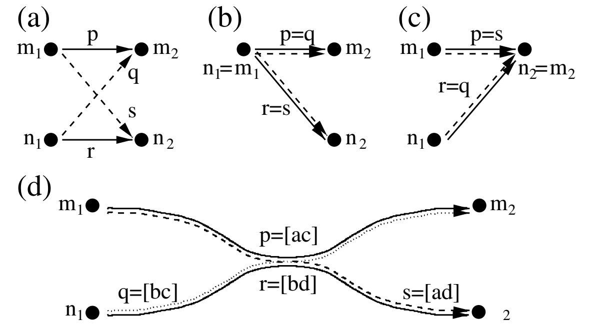

FIG. 2. (a) Shown schematically are four classical trajectories p, q, r, s connecting two incoming modes m_1, n_1 to two outgoing modes m_2, n_2 such that the diagram contributes to the semiclassical approximation of Eq. (2). A contribution to Eq. (6) results only if the trajectories are correlated such that the action difference between p, r (solid lines) and q, s (dashed lines) remains small for varying energy. (b),(c) The simplest configurations where this is the case: Two trajectories are pairwise equal (diagonal approximation). (d) The configuration which completely accounts for the universal shot noise in leading order.

For shot noise, the diagonal approximation has two analogues [Figs. 2(b) and 2(c)]: We can have (i) $p = q$, $r = s$ for $m_1 = n_1$ and (ii) $p = s$, $r = q$ for $m_2 = n_2$. In both cases, no phases are left in Eq. (6) and the remaining summation is over two independent classical trajectories p, r with the only constraint that they begin or end at the same mode, respectively. Proceeding as in the diagonal approximation to the conductance, we obtain [30]

$$\sum_{m_1=1}^{N_1}\sum_{m_2,n_2=1}^{N_2}\sum_{pr} |A_p|^2|A_r|^2 = \frac{N_1 N_2^2}{(N_1 + N_2)^2}, \tag{7}$$

$$\sum_{m_1,n_1=1}^{N_1}\sum_{m_2=1}^{N_2}\sum_{pr} |A_p|^2|A_r|^2 = \frac{N_1^2 N_2}{(N_1 + N_2)^2}. \tag{8}$$

Combining these two results, we find

$$\langle \mathrm{Tr}\,(tt^\dagger)^2\rangle_{\mathrm{diag}} = \langle \mathrm{Tr}\, tt^\dagger\rangle_{\mathrm{diag}} = N_1 N_2/(N_1 + N_2), \tag{9}$$

and, according to Eq. (2), this means that within the diagonal approximation there is no shot noise. This is no surprise: The diagonal approximation reduces the quantum to the classical problem and, since classical dynamics is deterministic, there is no uncertainty if an incoming electron is transmitted or not and, hence, no noise [14,31]. On the other hand, Eq. (9) is quite remarkable as it means that within the semiclassical approximation shot noise is entirely due to nontrivial correlations between different trajectories.

What is the general mechanism for such correlations? Previous research [9–13] showed that pairs of trajectories have correlated actions if they explore the same (or symmetry-related) parts of phase space with a different itinerary. In terms of symbolic dynamics, the code words for the two orbits are composed of the same sequences, in permuted order. The analogy to diagrammatic perturbation theory and some recent results [11,12] suggest further that the importance of the correlations decreases with a growing number of sequences needed to represent the code of the trajectories: In the diagonal approximation to the conductance, the codes are equal and the result is correct to leading order, orbit pairs composed of two loops give the next-to-leading order correction, etc.

In the case of shot noise, we have exhausted the diagonal approximation and consider therefore trajectories p, q, r, s whose codes can all be represented in terms of two subsequences. Inspection shows that the only option is the diagram of Fig. 2(d). Indeed the phase in Eq. (6) will almost vanish for such contributions since the combination of p, r (solid lines) almost coincides with the combination of q, s (dashed lines) such that the respective actions cancel. A remaining small total action difference comes from the different behavior of the trajectories inside the crossing region. In this respect, the correlated trajectories of Fig. 2(d) are very similar to those giving rise to weak localization effects [11–13]; i.e., the methods developed there for various specific systems should allow for a straightforward generalization to shot noise.

In the remainder of this Letter, we treat one of those systems explicitly, namely, the quantum graph shown in Fig. 1(b). The closed version of this graph consists of a central vertex with valency B and $b = 1 \cdots B$ attached bonds with incommensurate lengths L_b. Following the standard quantization [23], the dynamics of a particle with wave number $k = (2mE)^{1/2}/\hbar$ is represented by a $B \times B$ bond-scattering matrix $\Sigma_{bb'}(k) = \sigma_{bb'} e^{2ikL_b}$ containing energy-dependent phases $2kL_b$ from the free motion on the bonds and complex amplitudes $\sigma_{bb'}$ describing the scattering at the central vertex. A basic requirement is unitarity (current conservation)

$$\delta_{\alpha\alpha'} = \sum_{\beta=1}^{B} \sigma_{\beta\alpha}\sigma^*_{\beta\alpha'}, \qquad (10)$$

otherwise σ can be chosen according to the physical situation. For simplicity, we set

$$\sigma_{bb'} = e^{2\pi i bb'/B}/\sqrt{B}, \qquad (11)$$

such that all classical transition probabilities are equal $|\sigma_{bb'}|^2 \equiv 1/B$. This model was first considered by Tanner [25], who showed numerically that its spectral statistics follows RMT. We open the graph by extending $N = N_1 + N_2$ bonds to infinity and model a two-channel geometry by considering N_1 (N_2) of these leads as the modes in the left (right) contact [Fig. 1(b)]. For fixed N_1, N_2 we will consider the limit $B \to \infty$ in order to meet the condition of a long dwell time which was already mentioned in connection with the diagonal approximation. Within the leading order in B, we also neglect lower-order corrections in the mode numbers N_1, N_2 in order to compare our result to Eq. (3).

For a graph, the $N \times N$ unitary scattering matrix Eq. (1) can be expressed in terms of subblocks of the bond-scattering matrix Σ via $S = \Sigma_{\mathcal{L}\mathcal{L}} + \Sigma_{\mathcal{L}\mathcal{G}}(I - \Sigma_{\mathcal{G}\mathcal{G}})^{-1}\Sigma_{\mathcal{G}\mathcal{L}}$ [23]. $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ denotes here the set of $N = N_1 + N_2$ leads and $\mathcal{G}$ comprises the $B - N$ bonds inside the graph. Expanding the Green's function of the internal part $(I - \Sigma_{\mathcal{G}\mathcal{G}})^{-1}$ into a geometric series, we arrive at Eq. (4) which is in the case of graphs an identity rather than a semiclassical approximation. The sum is over all trajectories (= bond sequences) $p = [n_1 p_1 \cdots p_t n_2]$ connecting the lead $n_1 \in \mathcal{L}_1$ to the lead $n_2 \in \mathcal{L}_2$ via an arbitrary number $t \geq 0$ of internal bonds $p_j \in \mathcal{G}$. The action is related to the total length of the trajectory, $S_p/\hbar = kL_p$, where $L_p = \sum_{j=1}^{t} L_{p_j}$. Finally, the amplitude is given as $A_p = \sigma_{n_1 p_1}\sigma_{p_1 p_2} \cdots \sigma_{p_t n_2}$ such that the classical probability of the trajectory is $|A_p|^2 = 1/B^{t+1}$. Consequently, we have

$$\sum_p |A_p|^2 = \sum_{t=0}^{\infty} \frac{(B-N)^t}{B^{t+1}} = \frac{1}{N}, \qquad (12)$$

from which we do indeed recover the diagonal approximation Eqs. (7)–(9).

Next we consider trajectories p, q, r, s which are composed of four sequences a, b, c, d as shown in Fig. 2(d) and assume that each of these sequences has a length $t \geq 1$. In order to avoid overcounting, we have to ensure that for any given set p, q, r, s the definition of a, b, c, d is unique. Potential problems arise if all four trajectories coincide in the crossing region for one (or more) steps: $p = [a\gamma c]$, $q = [b\gamma c]$, $r = [b\gamma d]$, $s = [a\gamma d]$. It is a matter of taste if γ in these situations is considered as part of a and b or of c and d. We use the first representation and enforce it by the restriction $c_i \neq d_i$. (The subscripts i/f are used for the initial/final bond in a, b, c, d.)

Returning to Eq. (6), we note that the actions of p, r and q, s cancel exactly such that the phase factor is absent. As in Eq. (12), we can perform the summation over all internal bonds of the subsequences a, b, c, d and also over the leads m_1, n_1, m_2, n_2. Only the amplitudes from transitions right at the intersection of a, b, c, d do not combine into classical probabilities and must be considered explicitly. We obtain

$$t^{(4)}_{abcd} = \frac{N_1^2N_2^2}{N^4} \sum_{a_f b_f, c_i \neq d_i \in \mathcal{G}} \sigma_{a_f c_i}\sigma^*_{b_f c_i}\sigma_{b_f d_i}\sigma^*_{a_f d_i}$$

$$= \frac{N_1^2N_2^2}{N^4} \sum_{a_f b_f \in \mathcal{L}} \sum_{c_i d_i \in \mathcal{G}} \sigma_{a_f c_i}\sigma^*_{b_f c_i}\sigma_{b_f d_i}\sigma^*_{a_f d_i}(1-\delta_{c_i d_i}) \tag{13}$$

$$= \frac{N_1^2N_2^2}{N^4}\Bigg(\sum_{a_f = b_f \in \mathcal{L}} \sum_{c_i d_i \in \mathcal{G}} + \sum_{a_f \neq b_f, c_i d_i \in \mathcal{L}} - \sum_{a_f \neq b_f \in \mathcal{L}} \sum_{c_i = d_i \in \mathcal{G}} \Bigg) \sigma_{a_f c_i}\sigma^*_{b_f c_i}\sigma_{b_f d_i}\sigma^*_{a_f d_i} \tag{14}$$

$$= \frac{N_1^2N_2^2}{N^3} + \mathcal{O}(B^{-1}). \tag{15}$$

To perform this calculation, we have repeatedly used Eq. (10) in a form which allows one to transfer summations from the graph $\mathcal{G}$ to the leads $\mathcal{L}$

$$\sum_{\beta \in \mathcal{G}} \sigma_{\beta\alpha}\sigma^*_{\beta\alpha'} = \delta_{\alpha\alpha'} - \sum_{\beta \in \mathcal{L}} \sigma_{\beta\alpha}\sigma^*_{\beta\alpha'}. \tag{16}$$

The second and the third sums in Eq. (14) yield only a negligible correction $\mathcal{O}(B^{-1})$ because the number of terms are $N^3(N-1)$ and $N(N-1)(B-N)$, respectively, and $|\sigma_{a_f c_i}\sigma^*_{b_f c_i}\sigma_{b_f d_i}\sigma^*_{a_f d_i}| = B^{-2}$.

With exactly the same methods, we can consider the contribution from special cases of the diagram in Fig. 2(d) where the length of one of the subsequences a, b, c, d vanishes. We find that for vanishing a or b we get $t^{(4)}_{bcd} = t^{(4)}_{acd} = -t^{(4)}_{abcd}$ while for vanishing c or d no contribution results. Finally, we have $\langle \mathrm{Tr}\,(tt^\dagger)^2 \rangle = t^{(4)}_{abcd} + t^{(4)}_{bcd} + t^{(4)}_{acd} = -N_1^2N_2^2/N^3$ and substitution of this result into Eq. (2) shows that we have indeed reproduced the RMT result from correlated classical trajectories.

We have checked that our result remains unchanged if we substitute in Eq. (2) $\mathrm{Tr}\,tt^\dagger(1-tt^\dagger) = \mathrm{Tr}\,tt^\dagger rr^\dagger$. Quantum mechanically, this is just a consequence of unitarity, but within the semiclassical approach it is a nontrivial result since entirely different trajectories contribute and unitarity is restored only if all relevant correlations between them are properly accounted for.

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*Electronic address: holger@chaos.gwdg.de

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