

EDGE SWITCHING TRANSFORMATIONS OF QUANTUM GRAPHS — A SCATTERING APPROACH

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*To the memory of Michael Solomyak – a teacher,
a colleague and a friend*

ABSTRACT. Some elementary transformations of quantum graphs are discussed and their effects on the spectra of the Schrödinger operators are studied. In particular, the edge swapping operation is considered, where the lengths of two edges are interchanged, as well as switching, where the connectivity of the graph is modified by reconnecting two edges. Both transformations preserve the total length of the graphs. This problem was already studied at length and in generality in a previous paper. Here, it is addressed from a different viewpoint based on a scattering approach, yielding a trace formula for the difference between the spectral counting functions before and after the transformation.

§1. SETTING THE STAGE

Quantum graphs were introduced years ago [2, 3] and were defined and discussed in detail in review articles and books (see, e.g., [5, 6]). The subsequent paragraphs are intended to introduce the definitions and notation to be used throughout.

A quantum graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{L})$ on a set of vertices \mathcal{V} (with $|\mathcal{V}| = V$), edges \mathcal{E} (with $|\mathcal{E}| = E$) and a list of (positive and bounded) edge lengths \mathcal{L} (with $|\mathcal{L}| = E$), is defined topologically in terms of its V -dimensional adjacency matrix A , and metrically by endowing the edges with the standard metric and with the edge lengths \mathcal{L} . It will be assumed that the graph is simple $A_{i,j} \in \{0, 1\}$. The associated Schrödinger operator consists of the one-dimensional Laplacian (possibly magnetic) and a potential on each edge. It acts on

$$L^2(\mathcal{G}) := \bigoplus_{e \in \mathcal{E}} L^2(e, dx).$$

The boundary conditions ensuring that the Schrödinger operator is selfadjoint are expressed as linear relations between the limiting values of the wave functions and their first derivative at each vertex.

The spectrum of the Schrödinger operator $\{E_n\}_{n=1}^\infty$ is known to be discrete, bounded from below, and in the following it will be assumed to be ordered as a monotone nondecreasing sequence.

Spectral geometry for quantum graphs attempts to provide a link connecting the graph topology and metric with the spectrum. For the case of vanishing potentials, it is known [7] that if the edge lengths of the graph are not commensurate “one can hear the shape of the graph” — the topology and the lengths are uniquely determined by the spectrum. Questions about the sensitivity of the spectrum to modifications of the graphs connectivity, metric and boundary conditions were studied, e.g., in [6, 8, 9, 10, 11].

Key words and phrases. Schrödinger operator, quantum graph, length swapping, edge switching.

In a recent paper [1], we defined and studied in detail several elementary transformations of graphs, and their effect on the spectra. Here, for reasons which will become clear in the sequel, we shall focus on two transformations.

Length swapping: Given two edges with lengths $L_r, L_s \in \mathcal{L}$, the swapping transformation interchanges the lengths $L_r \leftrightarrow L_s$, leaving the rest untouched.

Edge switching: Given four vertices $r, s, u, v \in \mathcal{V}$ such that $A_{r,s} = A_{u,v} = 1$ but $A_{r,u} = A_{v,s} = 0$, the switching transformation changes the connectivity $A \rightarrow \tilde{A}$ so that $\tilde{A}_{r,s} = \tilde{A}_{u,v} = 0$ and $\tilde{A}_{r,u} = \tilde{A}_{v,s} = 1$. The newly generated edges are endowed with the lengths of the removed edges. A more general definition of swapping can be found in [16, 17].

Swapping does not alter the graph connectivity at all, while switching as defined above changes topology without changing the vertex degrees. Also note that the two transformations do not affect the total length of the graph.

The main result of [1], restricted to the swapping and switching transformations, is summarized in the following theorem.

Theorem 1. *Given two quantum graphs as described above that are different only by a single transformation, denote their spectra by $\{E_n\}$ and $\{\tilde{E}_n\}$. Then $\{E_n\}$ and $\{\tilde{E}_n\}$ interlace so that $E_{n-2} \leq \tilde{E}_n \leq E_{n+2}$ for $n > 2$.*

Remark 1.1. Denote by $N(E)$ and $\tilde{N}(E)$ the spectral counting functions before and after a switching. The result of Theorem 1 is equivalent to saying that

$$|N(E) - \tilde{N}(E)| \leq 2.$$

Remark 1.2. For transformations that involve edges connected by a vertex, the interlacing is more restricted, namely,

$$E_{n-1} \leq \tilde{E}_n \leq E_{n+1}.$$

See [1] for the details.

The method of proof used in [1] is based on the theory of low rank perturbations of selfadjoint operators [12, 13], which made possible the discussion of general Schrödinger operators on the edges, and allowed one to consider a large variety of elementary transformations. Similar approaches had been used before in [4, 14, 15].

Here, we would like to address the same problem from a different point of view, namely, we use the scattering approach to quantum graphs [2, 3]. Its main advantage is that it provides a convenient starting point for developing a periodic orbit theory which will be used to express the difference between the counting functions $N(E) - \tilde{N}(E)$ in a transparent way. Thus, the interlacing property, which bounds this function from below and above, may be extended by more detailed and quantitative information on the spectral difference between the transformed graphs.

§2. THE SCATTERING APPROACH

The scattering approach is based on expressing the vertex boundary conditions in terms of the unitary vertex scattering matrices $\sigma^{(v)}$, $v \in \mathcal{V}$, see [2, 3]. To simplify the presentation it will be assumed that the Schrödinger operators on all edges consist of the free Laplacians only, the wave functions are continuous across the vertices and satisfy mixed (Robin) boundary conditions that correspond to positive δ vertex potentials. In this case $E_n \geq 0$ for all n , and it is convenient to use the wave number $k = \sqrt{E}$ instead of the energy as a parameter. In particular, here we shall use the term “spectrum” for the sequence $k_n = |\sqrt{E_n}|$. The main result of the scattering approach is the theorem

saying that the k_n are the zeros of a secular function $\xi(k)$,

$$(1) \quad \begin{aligned} \xi(k) &= \det \left[I^{(2E)} - U(k) \right], \\ U(k) &= D(k)\Sigma(k). \end{aligned}$$

Here the matrices $I^{(2E)}$, $U(k)$, $D(k)$, and $\Sigma(k)$ are of dimension $2E$, and their indices refer to the directed edges obtained by assigning two directions on each edge. $I^{(2E)}$ is the unit matrix and $U(k)$ is the so-called evolution operator on the quantum graph, and it is defined as the product $D(k)\Sigma(k)$. $D(k)$ is a diagonal matrix with entries $D_{d,d}(k) = e^{ikL_d}$, $L_d \in \mathcal{L}$, where the lengths of a directed edge d and its reversed edge \hat{d} are equal. The matrix $\Sigma(k)$ is constructed from the vertex scattering matrices: given two directed edges d, d' , where d' terminates at a vertex v and d originates at v , we have

$$\Sigma_{d,d'}(k) = \sigma_{d,d'}^{(v)}(k).$$

Denote the spectrum of the unitary matrix $U(k)$ by $\{e^{i\theta_j(k)}\}_{j=1}^{2E}$, and define the phases $\theta_j(k)$ in the interval $(0, 2\pi]$. Equation (1) implies that the spectrum k_n is identified by the requirement that $\theta_j(k_n) = 0$ for some j . This intimate relationship between the spectrum of $U(k)$ on the unit circle and the spectrum of the Schrödinger operator will play an important role in what follows.

To introduce the scattering approach in the present context, first we apply it to the length swapping transformation, where the lengths $L_r, L_s \in \mathcal{L}$, ($r \neq s$) are swapped. This amounts to the permutation $L_r \leftrightarrow L_s$ in \mathcal{L} . Thus, the original and the resulting quantum graphs are topologically identical, however their spectra are different whenever the lengths \mathcal{L} are independent over the rationals [7] and if one excludes trivial cases such as, e.g., a Neumann vertex of degree 2 which divides an edge.

Considering the effect of swapping on the secular equation (1), we note that the length information appears only in the diagonal matrix $D(k)$, where four entries are altered because each length appears twice, once for each direction on both edges. Define two unitary matrices

$$(2) \quad V_{(+)} = I^{(2E)} - [1 - e^{ik(L_s - L_r)}](|r\rangle\langle r| + |\hat{r}\rangle\langle \hat{r}|)$$

where $|r\rangle$ (or $|\hat{r}\rangle$) denotes the indicator vectors, whose entries are all zero but for a single entry 1 at the position corresponding to the directed edges r (or \hat{r}) both having the same length L_r . Similarly,

$$(3) \quad V_{(-)} = I^{(2E)} - [1 - e^{-ik(L_s - L_r)}](|s\rangle\langle s| + |\hat{s}\rangle\langle \hat{s}|).$$

Denote by $\tilde{U}(k)$ the graph evolution operator after swapping. Then

$$(4) \quad \tilde{U}(k) = V_{(+)}V_{(-)}D(k)\Sigma(k) = V_{(+)}V_{(-)}U(k).$$

Note that the main difference between $V_{(+)}$ and $V_{(-)}$ is the sign of the phases in the term appearing in square brackets in (2) and (3).

The proof of Theorem 1 (for swapping) proceeds in two steps. The first step compares the spectra of the operators $U(k)$, and $\tilde{U}(k)$ on the unit circle, and it culminates in a lemma proving that they interlace: $\theta_{n-2} \leq \tilde{\theta}_n \leq \theta_{n+2}$ for all k . The second step completes the proof by showing that the interlacing of the spectra on the unit circle implies a similar interlacing of the spectra k_n and \tilde{k}_n on the real line.

An auxiliary lemma will be used in order to study the difference between the spectra of the evolution operators. Consider a unitary matrix U of dimension N , and an arbitrary

complex column vector $|v\rangle = (v_1, \dots, v_N)^\top$ and its Hermitian conjugate $\langle v|$ normalized so that $\langle v|v\rangle = 1$. Construct the unitary matrix

$$V = I^{(N)} - [1 - e^{i\alpha}] |v\rangle\langle v|,$$

where $I^{(N)}$ is the identity matrix in dimension N and α an arbitrary real in $[0, 2\pi]$. Denote the eigenvalues of U by $\{e^{i\theta_j}\}_{j=1}^N$, $\theta_j \in (0, 2\pi]$, and the spectrum of the product VU by $\{e^{i\phi_j}\}_{j=1}^N$, $\phi_j \in (0, 2\pi]$.

Lemma 2.1. *The spectra of VU and U interlace so that for $\alpha > 0$ we have $\theta_j \leq \phi_j \leq \theta_{j+1}$ and for $\alpha < 0$ we have $\theta_{j-1} \leq \phi_j \leq \theta_j$. (In the above, $\theta_0 = \theta_N - 2\pi$, $\theta_{N+1} = \theta_1 + 2\pi$.)*

Remark 2.1. The interlacing means that between any two neighboring eigenphases of U there is exactly one eigenphase of VU and *vice versa*. This includes the gap between θ_N and θ_1 (because of the cyclic definition for θ_0 and θ_{N+1} in the theorem). Therefore, the interlacing property does not depend on a shift of the domain of all phases, e.g., to $0 \leq \theta_i < 2\pi$.

Proof. The proof of Lemma 2.1 can be found, e.g., in [18] and the references therein. It is based on the observation that the new spectrum consists of the N solutions $\{\phi_j\}_{j=1}^N$ of the equation

$$(5) \quad \cot \frac{\alpha}{2} + \sum_{j=1}^N |v_j|^2 \cot \frac{\theta_j - \phi}{2} = 0.$$

From here the result follows directly. \square

V is referred to in the literature as a multiplicative rank-1 perturbation. This Lemma 2.1 is analogous to Weyl's lemma for additive rank-1 perturbations. It can be generalized to perturbations of any rank (smaller than N) by multiple application of rank-1 perturbations. Using Lemma 2.1, we can prove the next claim.

Lemma 2.2. *The spectra of $U(k)$ and of $\tilde{U}(k)$ interlace on the unit circle, with $\theta_{n-2} \leq \tilde{\theta}_n \leq \theta_{n+2}$ for all k .*

Proof. The matrix $V_{(+)}V_{(-)}$ can be written as a product of four multiplicative rank-1 perturbations. The phases appearing in the first two factors are complex conjugate to the last, and hence they induce interlacing of rank 2 in opposite directions when applied to $U(k)$. This completes the proof of Lemma 2.2. \square

At the second step, the spectrum of the evolution operator on the unit circle and the spectrum of the graph on the real line must be related to each other. For this, we apply and extend an earlier result by one of us ([4, (3.24)]).

Definition. For a unitary matrix $U(k)$ with smooth k -dependence, the Wigner–Smith operator [19] is

$$(6) \quad T(k) = \frac{1}{i} \frac{dU}{dk} U^\dagger$$

and its trace

$$\text{tr } T = \sum_{m=1}^M \frac{\partial \vartheta_m(k)}{\partial k}$$

is called the time delay of $U(k)$.

Remark 2.2. For $U = U_1 U_2$ the time delay is additive, $\text{tr } T = \text{tr } T_1 + \text{tr } T_2$. This is checked by a direct calculation using the invariance of the trace under cyclic permutations.

Lemma 2.3. *Let \mathcal{G} be a quantum graph and $N(k)$ its spectral counting function. Let $U(k)$ be a unitary matrix of size $M \times M$ with positive definite $T(k)$ such that the spectrum of the graph coincides with the solutions of the secular equation*

$$\det[I - U(k)] = 0.$$

Let

$$\{e^{i\vartheta_m(k)}\}_{m=1}^M, \quad \vartheta_m \in [0, 2\pi),$$

be the spectrum of $U(k)$ and

$$\Theta(k) = \sum_{m=1}^M \vartheta_m(k).$$

Then

$$(7) \quad \int_0^k dk \operatorname{tr} T(k) = 2\pi[N(k) - N(0)] + [\Theta(k) - \Theta(0)].$$

Proof. If $u_m(k)$ denotes the eigenvectors of $U(k)$, then

$$(8) \quad e^{i\vartheta_m(k)} = \langle u_m(k) | U(k) | u_m(k) \rangle,$$

$$(9) \quad \vartheta'_m(k) = \left\langle u_m(k) \left| \frac{1}{i} \frac{dU}{dk} U^\dagger \right| u_m(k) \right\rangle = \langle u_m(k) | T(k) | u_m(k) \rangle.$$

Thus, the positive definite $T(k)$ is equivalent to the strictly monotone increasing eigenphases of $U(k)$. Integrating (9) over $[0, k]$ we must account for the winding numbers of the phases $\vartheta(k)$: whenever a phase has completed a full winding around the unit circle and jumps from 2π to 0, there is a solution of the secular equation. On the other hand, if a phase is strictly monotone increasing, it must complete a full turn around the unit circle between two crossings of zero. Therefore, the sum of all winding numbers counts the number of eigenvalues of the graph in the integration interval. Summation over m on the right-hand-side of (9) yields the trace of $T(k)$. \square

Theorem 1 follows from Lemma 2.2 and Lemma 2.3:

Proof. The evolution operator U (see (1)) satisfies the assumptions of Lemma 2.3. In particular, in [4] it was shown that in this case $T(k)$ is positive definite for graphs with any Robin boundary conditions. For two graphs with evolution operators U, \tilde{U} related as in (4), we have

$$\operatorname{tr} \tilde{T} = \operatorname{tr} T + \operatorname{tr} T_{(+)} + \operatorname{tr} T_{(-)} = \operatorname{tr} T,$$

which follows from (2) and (3). Moreover, in this case $\tilde{U}(0) = U(0)$ and thus $\tilde{\Theta}(0) = \Theta(0)$ and $\tilde{N}(0) = N(0)$. Applying this to (7), we get

$$(10) \quad \tilde{N}(k) - N(k) = (2\pi)^{-1}[\Theta(k) - \tilde{\Theta}(k)].$$

From the interlacing of the eigenphases, Lemma 2.2, we have

$$\tilde{\vartheta}_i(k) \geq \vartheta_{i-2}(k)$$

and summation for $i > 2$ yields

$$\Theta(k) - \tilde{\Theta}(k) \leq \vartheta_{2E} + \vartheta_{2E-1} - \tilde{\vartheta}_2 - \tilde{\vartheta}_1 \leq 4\pi.$$

The last inequality follows because the phases are restricted to $[0, 2\pi)$. Similar arguments yield the lower bound -4π . Hence, by (10),

$$(11) \quad -2 \leq \tilde{N}(k) - N(k) \leq 2. \quad \square$$

Figure 1 illustrates the effect of swapping on the secular and the spectral counting functions.

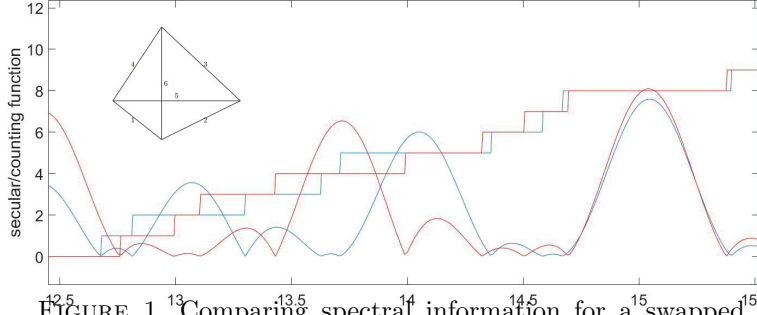


FIGURE 1. Comparing spectral information for a swapped pair in a complete $V = 4$ graph (shown in the insert) for the finite spectral interval $k \in [4\pi, 5\pi]$. On display we see the secular functions $|\xi(k)|$ (1) whose zeros mark the spectral points and the spectral counting functions $N(k) - N(k = 4\pi)$ (Neumann boundary conditions were used.).

2.1. Another construction. An alternative form of the secular equation will be presented and its advantages for the study of elementary transformations will be highlighted. In particular, it provides a secular equation in which both the swapping and switching can be treated within the same construction. Moreover, the interchanged lengths appear separately in the resulting secular equation. This is not the case with the secular equation (1), where all the edges on the graph are treated on the same footing. Suppose that

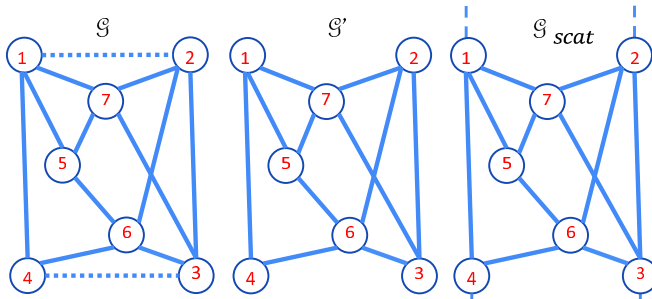


FIGURE 2. An illustration of three graphs \mathcal{G} , \mathcal{G}' , and $\mathcal{G}_{\text{scat}}$. The edges to be removed from \mathcal{G} are dotted, and their replacing leads to infinity are dashed.

the lengths on the edges labeled by r, s are to be swapped in the graph \mathcal{G} . Assume first that r, s have no common vertex. Denote by 1, 2 the vertices connected by the edge r , and by 3, 4 the vertices connected by the edge s . Consider now a new graph \mathcal{G}' constructed by removing from \mathcal{G} the edges r and s (see Fig. (2)). A scattering graph $\mathcal{G}_{\text{scat}}$ is now constructed by adding four leads to infinity, one for each of the vertices 1–4. The new graph $\mathcal{G}_{\text{scat}}$ is not compact, however all the vertex scattering matrices $\sigma^{(v)}$ are the same as in the original graph \mathcal{G} . One can use the methods developed in [21, 22, 23] to compute the unitary scattering matrix $S(k)$ of size 4×4 that provides the 4 amplitudes of the outgoing waves when 4 incoming waves impinge on $\mathcal{G}_{\text{scat}}$ from the 4 leads. Its dependence on k is complicated because it accounts for the possibly complex interference

patterns within \mathcal{G}' . Note however that $S(k)$ depends on the list of edge lengths \mathcal{L}' , which is the original list \mathcal{L} from which L_r, L_s are excluded. The eigenphases of $S(k)$ move monotonically counterclockwise along the unit circle as k increases, see [23].

The secular equation. Consider an arbitrary eigenstate of \mathcal{G} with wave number k . On the edge r it can be written as a superposition of two plane waves from vertex 1 to vertex 2 and back, that is,

$$\psi_r(x_{r,1}) = a_1^{(+)} e^{+ikx_{r,1}} + a_1^{(-)} e^{-ikx_{r,1}},$$

or alternatively,

$$\psi_r(x_{r,2}) = a_2^{(+)} e^{+ikx_{r,2}} + a_2^{(-)} e^{-ikx_{r,2}}.$$

The coordinate $x_{r,1}$ is the distance from vertex 1, while $x_{r,2}$ is measured from vertex 2, i.e.,

$$x_{r,1} = L_r - x_{r,2}.$$

On edge s we define equivalently the amplitudes $a_3^{(+)}, a_3^{(-)}, a_4^{(+)}, a_4^{(-)}$. Now, by definition, the scattering matrix $S(k)$ maps the coefficients of the incoming waves at the four vertices to the coefficients of the outgoing waves,

$$\mathbf{a}^{(+)} = S(k)\mathbf{a}^{(-)}.$$

On the other hand, the incoming waves are obtained from outgoing waves by simple phase shifts

$$\mathbf{a}^{(-)} = C(k)\mathbf{a}^{(+)}$$

with

$$(12) \quad C(k) = \begin{pmatrix} 0 & e^{ikL_r} & 0 & 0 \\ e^{ikL_r} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{ikL_s} \\ 0 & 0 & e^{ikL_s} & 0 \end{pmatrix}.$$

Putting both relations together, we have

$$\mathbf{a}^{(+)} = C(k)S(k)\mathbf{a}^{(+)},$$

which means that either the four coefficients $\mathbf{a}^{(+)}$ are zero, or

$$U_{\text{scat}}(k) = C(k)S(k)$$

has an eigenvalue one with eigenstate $\mathbf{a}^{(+)}$. We say that the eigenstates of \mathcal{G} with $\mathbf{a}^{(+)} = 0$ are *invisible*. (A full discussion of the S matrix and the possible appearance of eigenstates embedded in the continuum that are the invisible states can be found, e.g., in [23]). For our purpose they are irrelevant, because it is clear that such states do not change under a transformation that does not affect \mathcal{G}' . In other words, they do not affect the difference between the counting functions of the compared systems. For the visible eigenstates now we can state the secular equation

$$(13) \quad \begin{aligned} \xi_{\text{scat}}(k) &= \det \left[I^{(4)} - U_{\text{scat}}(k) \right]; \\ U_{\text{scat}}(k) &= C(k)S(k). \end{aligned}$$

It is equivalent to (1), but with an evolution operator which is simply a (4×4) -matrix. It can be viewed as an evolution operator of a much smaller graph consisting of a single vertex of degree 4 connected to two loops of lengths L_r and L_s formed by connecting the leads 1, 2 (or 3, 4) together (see Fig. 3(a)). The boundary conditions at this vertex are described by a vertex scattering matrix, which is $S(k)$ defined above and has a very complicated energy dependence accounting for the evolution in the hidden subgraph \mathcal{G}' .

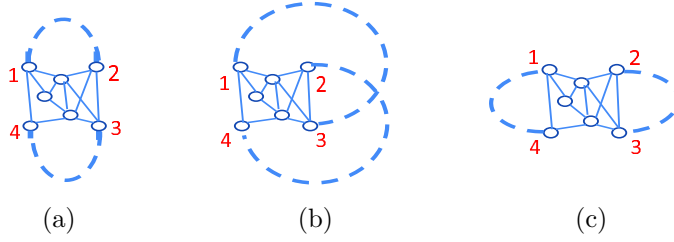


FIGURE 3. Three possible reconstructions: (a) the original graph or its swapped counterparts; (b,c) two possible switches.

The spectrum of the swapped graph consists of the zeros of $\xi_{\text{scat}}(k)$ where L_r and L_s are swapped in (12). When the edges (r, s) have a common vertex, a simple modification of the above discussion is necessary, but it will be omitted here for lack of space.

Edge switching can easily be incorporated in the formalism discussed in the preceding subsection by using the same construction, with the only difference that the spectrum of the quantum graph resulting after switching (see Figure 3 (b,c)) consists of the zeros of (13), where now

$$(14) \quad C(k) = \begin{pmatrix} 0 & 0 & e^{ikL_r} & 0 \\ 0 & 0 & 0 & e^{ikL_s} \\ e^{ikL_r} & 0 & 0 & 0 \\ 0 & e^{ikL_s} & 0 & 0 \end{pmatrix}.$$

This observation can be used to extend the proof given above to include not only edge swapping but also switching. For this, we can start with the secular equation (13), so that either switching or length swapping modifies the unitary matrix $U_{\text{scat}}(k)$ by the successive applications of two *multiplicative perturbations* of rank 2 with opposite phases. So by analogy with Lemma 2.2, the eigenphases of $U_{\text{scat}}(k)$ interlace in the range of two spectral intervals in both directions. Next we observe that also Lemma 2.3 remains valid. It was shown in [23] that the time-delay matrix (6) is positive definite for $S(k)$, i.e., for scattering from graphs with mixed boundary conditions. For $C(k)$ this can be shown by a direct calculation and then it also follows for their product $U_{\text{scat}}(k)$. Finally, (10) remains valid because again $\tilde{U}(0) = U(0)$. So the proof at the end of the previous section is essentially unchanged, simply the dimension of the operator $U(k)$ is 4 instead of $2E$.

§3. TRACE FORMULA

Trace formulas can be extracted from secular equations of type (13) in a standard way (see, e.g., [2, 3]). The graphs to be discussed consist of a single vertex carrying a vertex scattering matrix $\Sigma(k)$ and two loops connected to it. The periodic orbits have lengths $n_r L_r + n_s L_s$, where n_r, n_s are nonnegative integers counting the number of traversals of the corresponding loops. The trace formula will take the form

$$(15) \quad N(k) = N_{\Sigma}(k) + \frac{1}{\pi} \text{Im} \left\{ \sum_{n_r, n_s} A_{r,s}(\Sigma; k) e^{ik(n_r L_r + n_s L_s)} \right\}.$$

The swapped or switched lengths appear separated from all other lengths hidden in the coefficients $A_{r,s}(\Sigma; k)$. $N(k)$ for the original and transformed graphs have a common first term N_{Σ} and involve the same vertex matrix $\Sigma(k)$ to construct the coefficients $A_{r,s}(\Sigma; k)$. This facilitates appreciably the comparison between the corresponding spectra. Indeed,

since

$$\tilde{N}_\Sigma(k) = N_\Sigma(k),$$

we have

$$(16) \quad \tilde{N}(k) - N(k) = \frac{1}{\pi} \operatorname{Im} \left\{ \sum_{n_r, n_s} (\tilde{A}_{r,s}(\Sigma; k) - A_{r,s}(\Sigma; k)) e^{ik(n_r L_r + n_s L_s)} \right\}.$$

The above equation describes a piecewise constant function with zero mean and fluctuations bounded by 2. It can be used for the computation of the value distribution function of $\tilde{N}(k) - N(k)$ and its energy correlation function. Such computations are now under way.

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